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Constant terms in threshold resummation and the quark form factor

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Abstract: We verify to order α_s^4 two previously conjectured relations, valid in four dimensions, between constant terms in threshold resummation (for Deep Inelastic Scattering and the Drell-Yan process) and the second logarithmic derivative of the massless quark form factor. The same relations are checked to all orders in the large-β_0 limit; as a by-product a dispersive representation of the form factor is obtained. These relations allow to compute in a symmetrical way the three-loop resummation coefficients B_3 and D_3 in terms of the three-loop contributions to the virtual diagonal splitting function and to the quark form factor, confirming results obtained in the literature.

Keywords: QCD, NLO Computations.

Contents

1. Introduction

Threshold resummation, namely the resummation to all orders of perturbation theory of the large logarithmic corrections which arise from the incomplete cancellation of soft and collinear gluons at the edge of phase space, is by now a well developed subject [1, 2] in perturbative QCD. Large logarithms are however always accompanied by constant terms, whose contribution may be numerically important. In recent years, there has been some interest in the way these constant terms organize themselves. In [3], a relation between the constant terms and the massless quark form factor, valid in four dimensions, was conjectured in the case of Deep Inelastic Scattering (DIS) and Drell-Yan (DY). In this paper, we check this conjecture to order α_s^4. In addition, a check to all orders is performed in the large-n_f limit, based on the dispersive approach.
The paper is organized as follows. The order α_s^4 check is performed in sections 2 and 3 for DIS and DY, respectively. In section 4, we point out that the conjecture, if correct, is

valid for the most general class of resummation procedures. The all-order large-n_f check is performed in section 5, where contact is made with the dispersive approach [3, 4] viewed as a peculiar resummation procedure; as a by-product, a dispersive representation of the large-n_f quark form factor is obtained. Section 6 contains our conclusions. More technical issues are dealt with in three appendices. In appendix A an original method, based on the Mellin-Barnes representation, is exposed to compute the large-N (index) behavior of moments of +-distributions, including constant terms. In appendix B, the large-N scaling behavior of the characteristic functions which occur in the dispersive approach is derived. In appendix C the calculation of the massless one-loop quark form factor with a finite gluon mass, which gives the characteristic function of the large-n_f quark form factor, is detailed. The method we use is based on the resummation of the small gluon mass asymptotic expansion, itself derived through the Mellin-Barnes representation technique.

2. Threshold resummation of the physical anomalous dimension (DIS case)

We recall the standard resummation formula [1, 2, 5]. Consider the leading (twist 2) contribution to the non-singlet structure function $F_2(Q^2, N)$ in Mellin N-space

$$F_2(Q^2, N) = O(N, \mu^2)\ \mathcal{C}(Q^2, N, \mu^2)\,, \tag{2.1}$$

where $O(N, \mu^2)$ is the matrix element, $\mathcal{C}(Q^2, N, \mu^2)$ the coefficient function, and μ^2 the factorization scale. At large-N we have, neglecting terms that fall as $1/N$ (up to logarithms) order by order in perturbation theory

$$\mathcal{C}(Q^2, N, \mu^2) \sim g_{\rm DIS}(Q^2, \mu^2)\ \exp[E_{\rm DIS}(Q^2, N, \mu^2)]\,, \tag{2.2}$$

with the Sudakov exponent given by

$$E_{\rm DIS}(Q^2, N, \mu^2) = \int_0^1 dz \frac{z^{N-1}-1}{1-z}\left[\int_{\mu^2}^{(1-z)Q^2} \frac{dk^2}{k^2} A\left(a_s(k^2)\right) + B\left(a_s((1-z)Q^2)\right)\right]\,, \tag{2.3}$$

where, following the conventions of [5], with $a_s \equiv \frac{\alpha_s}{4\pi}$

$$A(a_s) = \sum_{i=1}^{\infty} A_i a_s^i \tag{2.4}$$

(A is the universal "cusp" anomalous dimension), and

$$B(a_s) = \sum_{i=1}^{\infty} B_i a_s^i\,, \tag{2.5}$$

are the usual Sudakov anomalous dimensions, whereas

$$g_{\rm DIS}(Q^2, \mu^2) = 1 + \sum_{i=1}^{\infty} g_i^{\rm DIS}\left(\frac{Q^2}{\mu^2}\right) a_s^i(\mu^2) \tag{2.6}$$

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collects the residual constant (N-independent) terms not included in $E_{\rm DIS}$. We note that $g_{\rm DIS}$ is *different* from g_0 as defined in [5] (which collects *all* the constant terms on the right-hand side of eq. (2.2)). Taking the derivative of eq. (2.3) we get [6, 7]

$$\frac{dE_{\rm DIS}(Q^2,N,\mu^2)}{d\ln Q^2}=\int_0^1 dz\frac{z^{N-1}-1}{1-z}\mathcal{J}[(1-z)Q^2]\,, \tag{2.7}$$

where

$$\mathcal{J}(k^2)=A\left(a_s(k^2)\right)+\frac{dB\left(a_s(k^2)\right)}{d\ln k^2}\,, \tag{2.8}$$

and $\mathcal{J}$ refers to the "jet scale" $(1-z)Q^2$ in eq. (2.7). Thus for the "physical anomalous dimension" [8, 9] which describes the scaling violation one obtains at large-N

$$\frac{d\ln F_2(Q^2,N)}{d\ln Q^2}=\frac{d\ln\mathcal{C}(Q^2,N,\mu^2)}{d\ln Q^2}\sim\int_0^1 dz\frac{z^{N-1}-1}{1-z}\mathcal{J}(1-z)Q^2]+H\left(a_s(Q^2)\right)\,, \tag{2.9}$$

where

$$H\left(a_s(Q^2)\right)=\frac{d\ln g_{\rm DIS}(Q^2,\mu^2)}{d\ln Q^2}\,. \tag{2.10}$$

Both the "Sudakov effective coupling"

$$\mathcal{J}(k^2)=\sum_{i=1}^{\infty}\mathcal{J}_i a_s^i(k^2) \tag{2.11}$$

and the "leftover" constant terms function

$$H(a_s)=\sum_{i=1}^{\infty}H_i a_s^i \tag{2.12}$$

are renormalization group invariant quantities, given as power series in a_s. At the difference of the usual Sudakov "anomalous dimensions" A and B, they are also renormalization scheme independent quantities.
Changing variables to $k^2=(1-z)Q^2$, eq. (2.7) becomes identically

$$\frac{dE_{\rm DIS}(Q^2,N,\mu^2)}{d\ln Q^2}=\int_0^{Q^2}\frac{dk^2}{k^2}F_{\rm DIS}\left(\frac{k^2}{Q^2},N\right)\mathcal{J}(k^2)\,, \tag{2.13}$$

with

$$F_{\rm DIS}\left(\frac{k^2}{Q^2},N\right)=\left(1-\frac{k^2}{Q^2}\right)^{N-1}-1\,. \tag{2.14}$$

It was shown in [10] that, up to terms which vanish for $N\to\infty$, we have

$$\begin{aligned}\frac{dE_{\rm DIS}(Q^2,N,\mu^2)}{d\ln Q^2}&\sim\int_0^{Q^2}\frac{dk^2}{k^2}G_{\rm DIS}\left(\frac{Nk^2}{Q^2}\right)\mathcal{J}(k^2)\\&\equiv S_{\rm DIS}(Q^2,N)\,,\end{aligned} \tag{2.15}$$

with

$$G_{\rm DIS}\left(\frac{Nk^2}{Q^2}\right)=\exp\left(-\frac{Nk^2}{Q^2}\right)-1\,, \tag{2.16}$$

where $G_{\rm DIS}(Nk^2/Q^2)$ is obtained by taking the $N \to \infty$ limit of $F_{\rm DIS}(k^2/Q^2, N)$ with Nk^2/Q^2 fixed. Thus we obtain at large-N

$$\frac{d \ln F_2(Q^2,N)}{d \ln Q^2} \sim S_{\rm DIS}(Q^2,N) + H\left(a_s(Q^2)\right) \ . \tag{2.17}$$

An additional simplification is achieved by extending to infinity the upper limit of integration in eq. (2.15), and introducing a suitable UV subtraction term, thus obtaining, up to terms which vanish for $N \to \infty$

$$\begin{aligned} S_{\rm DIS}(Q^2,N) &\sim \int_0^\infty \frac{dk^2}{k^2} G_{\rm DIS}\left(\frac{Nk^2}{Q^2}\right) \mathcal{J}(k^2) - G_{\rm DIS}(\infty) \int_{Q^2}^\infty \frac{dk^2}{k^2} \mathcal{J}(k^2) \\ &= \int_0^\infty \frac{dk^2}{k^2} G_{\rm DIS}\left(\frac{Nk^2}{Q^2}\right) \mathcal{J}(k^2) + \int_{Q^2}^\infty \frac{dk^2}{k^2} \mathcal{J}(k^2)\,, \end{aligned} \tag{2.18}$$

where the (UV finite) combination of the two (separately UV divergent, but IR finite) integrals on the right-hand side, when expanded in powers of $a_s(Q^2)$, contains only logarithmic and constant terms, and is free of $\mathcal{O}\left(\frac{\ln^p N}{N}\right)$ terms (at the difference of the left-hand side). In the second line, we used that $G_{\rm DIS}(\infty) = -1$, corresponding to the virtual contribution (the -1 on the right-hand side of eq. (2.16)). Thus we have

$$\int_0^\infty \frac{dk^2}{k^2} G_{\rm DIS}\left(\frac{Nk^2}{Q^2}\right) \mathcal{J}(k^2) + \int_{Q^2}^\infty \frac{dk^2}{k^2} \mathcal{J}(k^2) = \sum_{i=1}^\infty \gamma_i(N)\ a_s^i(Q^2)\,, \tag{2.19}$$

with $(L \equiv \ln N)$

$$\begin{aligned} \gamma_1(N) &= \gamma_{11} L + \gamma_{10} \\ \gamma_2(N) &= \gamma_{22} L^2 + \gamma_{21} L + \gamma_{20} \\ \gamma_3(N) &= \gamma_{33} L^3 + \gamma_{32} L^2 + \gamma_{31} L + \gamma_{30} \\ &\textit{etc.} \ \ . \end{aligned} \tag{2.20}$$

Alternatively, one may remove the virtual contribution from the Sudakov integral (so that it contains only real gluon emission contributions), and merge it together with the "leftover" constant terms, which yields the equivalent result, in terms of two separately IR divergent (but UV finite) integrals

$$S_{\rm DIS}(Q^2,N) \sim \int_0^\infty \frac{dk^2}{k^2} \left[G_{\rm DIS}\left(\frac{Nk^2}{Q^2}\right) + 1\right] \mathcal{J}(k^2) - \int_0^{Q^2} \frac{dk^2}{k^2} \mathcal{J}(k^2)\ . \tag{2.21}$$

Using eq. (2.18) into eq. (2.17), we end up with the large-N expression

$$\frac{d \ln F_2(Q^2,N)}{d \ln Q^2} \sim \int_0^\infty \frac{dk^2}{k^2} G_{\rm DIS}\left(\frac{Nk^2}{Q^2}\right) \mathcal{J}(k^2) + \left[H\left(a_s(Q^2)\right) + \int_{Q^2}^\infty \frac{dk^2}{k^2} \mathcal{J}(k^2)\right]. \tag{2.22}$$

If instead one uses eq. (2.21) into eq. (2.17) one gets the equivalent form

$$\frac{d \ln F_2(Q^2,N)}{d \ln Q^2} \sim \int_0^\infty \frac{dk^2}{k^2} \left[G_{\rm DIS}\left(\frac{Nk^2}{Q^2}\right) + 1\right] \mathcal{J}(k^2) + \left[H\left(a_s(Q^2)\right) - \int_0^{Q^2} \frac{dk^2}{k^2} \mathcal{J}(k^2)\right]. \tag{2.23}$$

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Next we observe that the UV (respectively IR) divergences present in the individual integrals in eq. (2.22) (respectively eq. (2.23)) disappear after taking one more derivative (which eliminates the virtual contribution inside the Sudakov integral), namely

$$\frac{d^2 \ln F_2(Q^2,N)}{(d\ln Q^2)^2} \sim \int_0^\infty \frac{dk^2}{k^2}\dot{G}_{\rm DIS}\left(\frac{Nk^2}{Q^2}\right)\mathcal{J}(k^2) + \left[\frac{dH}{d\ln Q^2} - \mathcal{J}(Q^2)\right], \tag{2.24}$$

where $\dot{G}_{\rm DIS} \equiv -dG_{\rm DIS}/d\ln k^2$, and the integral in eq. (2.24)

$$S^{'}_{\rm DIS}\left(\frac{Q^2}{N}\right) \equiv \int_0^\infty \frac{dk^2}{k^2}\dot{G}_{\rm DIS}\left(\frac{Nk^2}{Q^2}\right)\mathcal{J}(k^2), \tag{2.25}$$

which depends on the *single* variable Q^2/N (the moment space "jet scale") is finite. In [3] it was conjectured that the combination $dH/d\ln Q^2 - \mathcal{J}(Q^2)$, which represents the "left-over" constant terms not included in $S^{'}_{\rm DIS}(Q^2/N)$, is related to the space-like on-shell electromagnetic massless quark form factor [12] $\mathcal{F}(Q^2)$ by the identity

$$\frac{d^2 \ln\left(\mathcal{F}(Q^2)\right)^2}{(d\ln Q^2)^2} = \frac{dH}{d\ln Q^2} - \mathcal{J}(Q^2)\ . \tag{2.26}$$

If this conjecture is correct, the second line of eq. (2.23), which involves an IR divergent integral, could be formally identified to the first logarithmic derivative of the square of the quark form factor, an IR divergent quantity:

$$\frac{d \ln\left(\mathcal{F}(Q^2)\right)^2}{d\ln Q^2} = H\left(a_s(Q^2)\right) - \int_0^{Q^2} \frac{dk^2}{k^2}\mathcal{J}(k^2)\ . \tag{2.27}$$

In this section, we show that eq. (2.26) can be checked to order a_s^4, using results in the litterature [5, 13, 14].

Let us first consider the left-hand side of eq. (2.26). We begin from the evolution equation satisfied [12] by the form factor in $D = 4-\epsilon$ dimensions (after multiplication by a factor of 2):

$$\frac{d \ln\left(\mathcal{F}(Q^2,\epsilon)\right)^2}{d\ln Q^2} = K\left(a_s(\mu^2),\epsilon\right) + G\left(\frac{Q^2}{\mu^2}, a_s(\mu^2),\epsilon\right), \tag{2.28}$$

where $K\left(a_s(\mu^2),\epsilon\right)$ is a counterterm function which contains only poles in $1/\epsilon$, and is independent of Q^2, while $G\left(\frac{Q^2}{\mu^2}, a_s(\mu^2),\epsilon\right)$ is *finite* in four dimensions. Taking a second derivative and letting $\epsilon = 0$ yields

$$\frac{d^2 \ln\left(\mathcal{F}(Q^2)\right)^2}{(d\ln Q^2)^2} = \frac{d}{d\ln Q^2}G\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) = -\mu^2\frac{\partial}{\partial\mu^2}G\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right)\ . \tag{2.29}$$

We now use the renormalization group equation satisfied [12] by the G function (at $\epsilon = 0$)

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \beta\left(a_s\right)\frac{\partial}{\partial a_s}\right)G\left(\frac{Q^2}{\mu^2}, a_s\right) = A\left(a_s\right) \tag{2.30}$$

to find

$$\frac{d^2 \ln \left(\mathcal{F}(Q^2)\right)^2}{(d \ln Q^2)^2} = \beta\left(a_s(\mu^2)\right) \frac{\partial}{\partial a_s} G\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) - A\left(a_s(\mu^2)\right) , \tag{2.31}$$

where the beta function is given by the series

$$\beta(a_s) = -\sum_{i=0}^{\infty} \beta_i a_s^{i+2} . \tag{2.32}$$

Now, since $\frac{d^2 \ln\left(\mathcal{F}(Q^2)\right)^2}{d(\ln Q^2)^2}$ is a renormalisation group invariant quantity, we can set $\mu = Q$, and get

$$\frac{d^2 \ln \left(\mathcal{F}(Q^2)\right)^2}{(d \ln Q^2)^2} = \beta\left(a_s(Q^2)\right) \frac{\partial}{\partial a_s} G\left(1, a_s(Q^2)\right) - A\left(a_s(Q^2)\right) . \tag{2.33}$$

This relation is useful because it will give us the possibility to use the expressions for G quoted in [13] for $\mu = Q$. Rewriting eq. (2.26) as

$$\mathcal{J}(Q^2) = -\frac{d^2 \ln \left(\mathcal{F}(Q^2)\right)^2}{(d \ln Q^2)^2} + \frac{dH}{d \ln Q^2} \tag{2.34}$$

and using

$$\frac{dH}{d \ln Q^2} = \beta\left(a_s(Q^2)\right) \frac{\partial}{\partial a_s} H\left(a_s(Q^2)\right) \tag{2.35}$$

the relation to be checked becomes

$$\mathcal{J}(Q^2) = A\left(a_s(Q^2)\right) + \beta\left(a_s(Q^2)\right) \frac{\partial}{\partial a_s}\left[-G\left(1, a_s(Q^2)\right) + H\left(a_s(Q^2)\right)\right] . \tag{2.36}$$

Therefore, comparing with eq. (2.8), we have to check that

$$B(a_s) = -G(1, a_s) + H(a_s) . \tag{2.37}$$

This is essentially a check of the "non-conformal" part[1] of $\mathcal{J}$. We note that a contribution at order a_s^i to B implies a contribution at order a_s^{i+1} to $\mathcal{J}$ in eq. (2.8). In particular, since B starts at order a_s, it will contribute to $\mathcal{J}$ only starting at order a_s^2, and the order a_s contribution will be entirely provided by the cusp anomalous dimension A, yielding $\mathcal{J}_1 = A_1$. Moreover, a check of eq. (2.37) to order a_s^i implies a check of eq. (2.36) to order a_s^{i+1}. Since the cusp anomalous dimension $A(a_s)$ does not appear explicitly in eq. (2.5), this observation implies that we shall be able to check eq. (2.26) to order a_s^4, despite the fact that $A(a_s)$ is only known to order[2] a_s^3. Following the conventions of [13] for the G function,

$$G(1, a_s) = \sum_{i=1}^{\infty} G_i a_s^i , \tag{2.38}$$

we therefore see that the proof of eq. (2.26) amounts to show that

$$G(1, a_s) = H(a_s) - B(a_s) , \tag{2.39}$$

[1] Eq. (2.37) also shows that the combination $G(1, a_s) + B(a_s)$ is renormalization scheme invariant.

[2] Actually, as we shall see, even A_3 is not needed up to this order.

i.e., for all $i \geq 1$,

$$G_i = H_i - B_i \ . \tag{2.40}$$

Let us come into the details of the proof for $i \leq 3$. The G_i's and B_i's have been computed in the litterature up to $i = 3$ but for the H_i's a bit of work is still needed. We have the following relation to determine H_i:

$$H\left(a_s(Q^2)\right) + C_{\rm DIS}\left(a_s(Q^2)\right) = \frac{d}{d\ln Q^2}\ln g_0^{\rm DIS}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) , \tag{2.41}$$

where $g_0^{\rm DIS}$ is a function[3] which collects *all* the constant terms on the right-hand side of eq. (2.2), and is thus *different* from $g_{\rm DIS}$, whereas $C_{\rm DIS}$ collects the constant terms included in the Sudakov integrals on the right-hand side of eq. (2.13) or (2.15). Eq. (2.41) simply expresses the fact that the constant terms on the right-hand side of eq. (2.17) are the sum of the constant terms originating from H and those included in the Sudakov integral $S_{\rm DIS}(Q^2, N)$. For $\mu = Q$ we have

$$g_0^{\rm DIS}(1, a_s) = 1 + \sum_{i=1}^{\infty} g_{0i}^{\rm DIS} \ a_s^i \, , \tag{2.42}$$

where the $g_{0i}^{\rm DIS}$'s are known [5] up to $i = 3$ and we show below that one can obtain $\frac{d}{d\ln Q^2}\ln g_0^{\rm DIS}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right)$ from $g_0^{\rm DIS}(1, a_s)$. On the other hand $C_{\rm DIS}$ is given by the series (see eq. (2.20))

$$C_{\rm DIS}(a_s) = \sum_{i=1}^{\infty} \gamma_{i0} \ a_s^i \ . \tag{2.43}$$

We begin with the calculation of the right-hand side of eq. (2.41). We have

$$\frac{d}{d\ln Q^2}\ln g_0^{\rm DIS}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) = -\mu^2\frac{\partial}{\partial\mu^2}\ln g_0^{\rm DIS}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) . \tag{2.44}$$

We can then use eq. (3.10) of [14] to obtain the renormalization group equation satisfied by $g_0^{\rm DIS}$:

$$\left[\mu^2\frac{\partial}{\partial\mu^2} + \beta\left(a_s(\mu^2)\right)\frac{\partial}{\partial a_s}\right]\ln g_0^{\rm DIS}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) = A\left(a_s(\mu^2)\right)\gamma_E - B_\delta\left(a_s(\mu^2)\right) , \tag{2.45}$$

where B_δ is the coefficient of $\delta(1-x)$ in the non-singlet splitting function. Its expansion in powers of a_s

$$B_\delta(a_s) = \sum_{i=1}^{\infty} B_i^\delta a_s^i \tag{2.46}$$

is known up to $i = 3$ and can be found in [14, 15]. We thus obtain

$$\begin{aligned}\frac{d}{d\ln Q^2}\ln g_0^{\rm DIS}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) &= -\left[A\left(a_s(\mu^2)\right)\gamma_E - B_\delta\left(a_s(\mu^2)\right)\right] \\ &\quad + \beta\left(a_s(\mu^2)\right)\frac{\partial}{\partial a_s}\ln g_0^{\rm DIS}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) .\end{aligned} \tag{2.47}$$

[3] $g_0^{\rm DIS}(1, a_s)$ is denoted $g_0(a_s)$ in [5].

Now, since $\frac{d}{d\ln Q^2}\ln g_0^{\rm DIS}\left(\frac{Q^2}{\mu^2},a_s(\mu^2)\right)$ is a renormalisation group invariant quantity, we can set $\mu^2=Q^2$ on the right-hand side of eq. (2.47) to get

$$\begin{aligned}\frac{d}{d\ln Q^2}\ln g_0^{\rm DIS}\left(\frac{Q^2}{\mu^2},a_s(\mu^2)\right) &= \left[B_\delta\left(a_s(Q^2)\right)-A\left(a_s(Q^2)\right)\gamma_E\right]\\ &\quad+\beta\left(a_s(Q^2)\right)\frac{\partial}{\partial a_s}\ln g_0^{\rm DIS}\left(1,a_s(Q^2)\right)\ . \end{aligned} \tag{2.48}$$

We then go on by computing the series coefficients in eq. (2.43) up to order a_s^3. Since $\mathcal{J}(k^2)$ is a renormalization group invariant effective charge, we have the well-known renormalization group logarithmic structure (see e.g. [8]), expanding in powers of $a_s(Q^2)$

$$\begin{aligned}\mathcal{J}(k^2) &= \mathcal{J}_1 a_s(Q^2)+\left[-\beta_0\mathcal{J}_1\ln\left(\frac{k^2}{Q^2}\right)+\mathcal{J}_2\right]a_s^2(Q^2)\\ &\quad+\left[\beta_0^2\mathcal{J}_1\ln^2\left(\frac{k^2}{Q^2}\right)-(\beta_1\mathcal{J}_1+2\beta_0\mathcal{J}_2)\ln\left(\frac{k^2}{Q^2}\right)+\mathcal{J}_3\right]a_s^3(Q^2)+\dots\ .\end{aligned} \tag{2.49}$$

Moreover eq. (2.8) gives:

$$\mathcal{J}_1 = A_1\,, \tag{2.50}$$

$$\mathcal{J}_2 = A_2-\beta_0 B_1 \tag{2.51}$$

and

$$\mathcal{J}_3 = A_3-\beta_1 B_1-2\beta_0 B_2\ . \tag{2.52}$$

Then the calculation of the coefficients γ_{i0} in eq. (2.43) up to $i=3$ is reduced to the evaluation of the constant terms c_p in the $N\to\infty$ asymptotic expansion of the integrals (for $p=0,1,2$):

$$I_p(N)\equiv\int_0^{Q^2}\frac{dk^2}{k^2}\left[\exp\left(-\frac{Nk^2}{Q^2}\right)-1\right]\ln^p\left(\frac{k^2}{Q^2}\right)\,, \tag{2.53}$$

because, as we have seen, $C_{\rm DIS}$ collects the constant terms of the right-hand side of eq. (2.15). The c_p can be obtained from standard results in the litterature (see e.g. [5]), since changing variable to $k^2/Q^2=1-z$, one gets

$$I_p(N)=\int_0^1 dz\frac{\exp[-N(1-z)]-1}{1-z}\ln^p(1-z)\ , \tag{2.54}$$

which was shown [16] to have the same large-N expansion (up to terms which vanish for $N\to\infty$) as $\bar{I}_p(N)\equiv\int_0^1 dz\frac{z^{N-1}-1}{1-z}\ln^p(1-z)$. One thus gets, for $p=0,1,2$:

$$c_0 = -\gamma_E\,, \tag{2.55}$$

$$c_1 = \frac{1}{2}\left(\gamma_E^2+\frac{\pi^2}{6}\right) \tag{2.56}$$

and

$$c_2=-\frac{1}{3}\left(\gamma_E^3+\gamma_E\frac{\pi^2}{2}+2\zeta_3\right)\ . \tag{2.57}$$

A novel derivation of these results is presented in appendix A. Using eq. (2.49), we then find the series in eq. (2.43) to be given by

$$C_{\rm DIS}(a_s) = \mathcal{J}_1 c_0 a_s + (-\beta_0 \mathcal{J}_1 c_1 + \mathcal{J}_2 c_0) a_s^2 + [\beta_0^2 \mathcal{J}_1 c_2 - (\beta_1 \mathcal{J}_1 + 2\beta_0 \mathcal{J}_2) c_1 + \mathcal{J}_3 c_0] a_s^3 + \dots \quad (2.58)$$

Now eqs. (2.41) and (2.48) yield

$$H(a_s) = B_\delta(a_s) - [C_{\rm DIS}(a_s) + A(a_s)\gamma_E] + \beta(a_s) \frac{\partial}{\partial a_s} \ln g_0^{\rm DIS}(1, a_s) \,. \quad (2.59)$$

For $i = 1$ one thus gets

$$H_1 = B_1^\delta - [A_1 \gamma_E + \mathcal{J}_1 c_0] = B_1^\delta \,, \quad (2.60)$$

where we used eq. (2.50). For $i = 2$ one gets

$$\begin{aligned} H_2 &= B_2^\delta - [A_2 \gamma_E + (\mathcal{J}_2 c_0 - \beta_0 \mathcal{J}_1 c_1)] - \beta_0 g_{01}^{\rm DIS} \\ &= B_2^\delta - \beta_0 \left[\gamma_E B_1 - A_1 c_1 + g_{01}^{\rm DIS}\right] \,, \end{aligned} \quad (2.61)$$

where we used eq. (2.51). Thus

$$H_2 = B_2^\delta + C_F \beta_0 (9 + 4\zeta_2) \,, \quad (2.62)$$

where we used [5]

$$A_1 = 4C_F \,, \quad (2.63)$$

$$B_1 = -3C_F \quad (2.64)$$

and

$$g_{01}^{\rm DIS} = C_F(-9 + 3\gamma_E + 2\gamma_E^2 - 2\zeta_2) \,. \quad (2.65)$$

Finally, for $i = 3$ we find

$$\begin{aligned} H_3 &= B_3^\delta - \left[A_3 \gamma_E + \left(\mathcal{J}_3 c_0 - (\beta_1 \mathcal{J}_1 + 2\beta_0 \mathcal{J}_2) c_1 + \beta_0^2 \mathcal{J}_1 c_2\right)\right] \\ &\quad - \left[\beta_1 g_{01}^{\rm DIS} + \beta_0 (2 g_{02}^{\rm DIS} - (g_{01}^{\rm DIS})^2)\right] \\ &= B_3^\delta - \beta_1 \left[\gamma_E B_1 - A_1 c_1 + g_{01}^{\rm DIS}\right] - \beta_0^2 \left[2 c_1 B_1 + A_1 c_2\right] \\ &\quad - \beta_0 \left[2\gamma_E B_2 - 2A_2 c_1 + 2 g_{02}^{\rm DIS} - (g_{01}^{\rm DIS})^2\right] \,, \end{aligned} \quad (2.66)$$

where we used eq. (2.52). We note that only A_j with $j < i$ contributes to H_i, and that the coefficient of the β_1 term in eq. (2.66) is the same as that of the β_0 term in eq. (2.61) (the general structure underlying these observations will be displayed below). Eq. (2.66) gives

$$\begin{aligned} H_3 &= B_3^\delta + 4C_F \beta_1 \left(\frac{\frac{9}{4} + \pi^2}{6}\right) + 4C_F \beta_0^2 \left(\frac{3}{4}\gamma_E^2 + \frac{1}{3}\gamma_E^3 + \frac{\pi^2}{8} + \gamma_E \frac{\pi^2}{6} + \frac{2}{3}\zeta_3\right) \\ &\quad + 4C_F \beta_0 \left[C_F \left(-\frac{7}{16} - \frac{25}{8}\pi^2 + \frac{\pi^4}{60} + 33\zeta_3\right)\right. \\ &\quad - C_A \left(-\frac{5465}{144} + \frac{11}{4}\gamma_E^2 + \frac{11}{9}\gamma_E^3 - \frac{469}{72}\pi^2 + \frac{11}{18}\gamma_E \pi^2 + \frac{71}{360}\pi^4 + \frac{232}{9}\zeta_3\right) \\ &\quad \left. + n_f \left(\frac{-457}{72} + \frac{1}{2}\gamma_E^2 + \frac{2}{9}\gamma_E^3 - \frac{35}{36}\pi^2 + \frac{1}{9}\gamma_E \pi^2 - \frac{2}{9}\zeta_3\right)\right] \,, \end{aligned} \quad (2.67)$$

where we used [5]

$$A_2 = C_F\left[\left(\frac{268}{9} - 8\zeta_2\right)C_A - \frac{40}{9}n_f\right], \tag{2.68}$$

$$B_2 = C_F C_A\left(-\frac{3155}{54}+\frac{44}{3}\zeta_2+40\zeta_3\right)+C_F n_f\left(\frac{247}{27}-\frac{8}{3}\zeta_2\right)-C_F^2\left(\frac{3}{2}-12\zeta_2+24\zeta_3\right) \tag{2.69}$$

and

$$\begin{aligned} g_{02}^{\rm DIS} = & \; C_F^2\left(\frac{331}{8} - \frac{51}{2}\gamma_E - \frac{27}{2}\gamma_E^2 + 6\gamma_E^3 + 2\gamma_E^4 + \frac{111}{2}\zeta_2 \right. \\ & \left. \quad - 18\gamma_E\zeta_2 - 4\gamma_E^2\zeta_2 - 66\zeta_3 + 24\gamma_E\zeta_3 + \frac{4}{5}\zeta_2^2\right) \\ & + C_F C_A\left(-\frac{5465}{72} + \frac{3155}{54}\gamma_E + \frac{367}{18}\gamma_E^2 + \frac{22}{9}\gamma_E^3 - \frac{1139}{18}\zeta_2 \right. \\ & \left. \quad - \frac{22}{3}\gamma_E\zeta_2 - 4\gamma_E^2\zeta_2 + \frac{464}{9}\zeta_3 - 40\gamma_E\zeta_3 + \frac{51}{5}\zeta_2^2\right) \\ & + C_F n_f\left(\frac{457}{36} - \frac{247}{27}\gamma_E - \frac{29}{9}\gamma_E^2 - \frac{4}{9}\gamma_E^3 + \frac{85}{9}\zeta_2 + \frac{4}{3}\gamma_E\zeta_2 + \frac{4}{9}\zeta_3\right). \end{aligned} \tag{2.70}$$

Substituting $n_f = -\frac{3}{2}\beta_0 + \frac{11}{2}C_A$ in the coefficient of the β_0 term in eq. (2.67) yields a further simplification:

$$\begin{aligned} H_3 = & \; B_3^\delta + C_F\beta_1(9+4\zeta_2) + C_F\beta_0^2\left(\frac{457}{12} + 38\zeta_2 + 4\zeta_3\right) \\ & + C_F\beta_0\left[C_F\left(-\frac{7}{4} - 75\zeta_2 + 132\zeta_3 + \frac{12}{5}\zeta_2^2\right) + C_A\left(\frac{73}{6} + 28\zeta_2 - 108\zeta_3 - \frac{142}{5}\zeta_2^2\right)\right]. \end{aligned} \tag{2.71}$$

We note that all γ_E terms, which arise entirely from the Sudakov integral (see eq. (2.58)), have cancelled in the H_i's.

We now have all ingredients to check eq. (2.40) for $i \le 3$ and verify eq. (2.26) order by order in a_s, up to order a_s^4. Since [14, 15]

$$B_1^\delta = 3C_F, \tag{2.72}$$

we get from eq. (2.60)

$$H_1 = 3C_F. \tag{2.73}$$

Moreover we have [13]

$$G_1 = 6C_F. \tag{2.74}$$

Thus we find, using eq. (2.64), that

$$G_1 = H_1 - B_1, \tag{2.75}$$

checking eq. (2.40) for $i = 1$. We next consider the case $i = 2$. Since [14, 15]

$$B_2^\delta = C_F C_A\left(\frac{17}{6} + \frac{44}{3}\zeta_2 - 12\zeta_3\right) - C_F n_f\left(\frac{1}{3} + \frac{8}{3}\zeta_2\right) + C_F^2\left(\frac{3}{2} - 12\zeta_2 + 24\zeta_3\right) \tag{2.76}$$

and

$$\beta_0 = \frac{11}{3} C_A - \frac{2}{3} n_f \,, \tag{2.77}$$

eq. (2.62) gives

$$H_2 = C_F C_A \left(\frac{215}{6} + \frac{88\zeta_2}{3} - 12\zeta_3 \right) - C_F n_f \left(\frac{19}{3} + \frac{16}{3}\zeta_2 \right) + C_F^2 \left(\frac{3}{2} - 12\zeta_2 + 24\zeta_3 \right) \,. \tag{2.78}$$

Moreover we have [13]

$$G_2 = C_F C_A \left(\frac{2545}{27} + \frac{44}{3}\zeta_2 - 52\zeta_3 \right) - C_F n_f \left(\frac{418}{27} + \frac{8}{3}\zeta_2 \right) + C_F^2 (3 - 24\zeta_2 + 48\zeta_3) \,. \tag{2.79}$$

We thus find, using eq. (2.69)

$$G_2 = H_2 - B_2 \,, \tag{2.80}$$

checking eq. (2.40) for $i = 2$. We also note the simple relations between the C_F^2 terms in eq. (2.69), (2.76), (2.78) and (2.79).

Let us finally consider the case $i = 3$. Since [14, 15]

$$\begin{aligned} B_3^\delta = & \; C_F^3 \left(\frac{29}{2} + 18\zeta_2 + 68\zeta_3 + \frac{288}{5}\zeta_2^2 - 32\zeta_2\zeta_3 - 240\zeta_5 \right) \\ & + C_F^2 C_A \left(\frac{151}{4} - \frac{410}{3}\zeta_2 + \frac{844}{3}\zeta_3 - \frac{988}{15}\zeta_2^2 + 16\zeta_2\zeta_3 + 120\zeta_5 \right) \\ & - C_F C_A^2 \left(\frac{1657}{36} - \frac{4496}{27}\zeta_2 + \frac{1552}{9}\zeta_3 + 2\zeta_2^2 - 40\zeta_5 \right) \\ & - C_F n_f^2 \left(\frac{17}{9} - \frac{80}{27}\zeta_2 + \frac{16}{9}\zeta_3 \right) - C_F^2 n_f \left(23 - \frac{20}{3}\zeta_2 + \frac{136}{3}\zeta_3 - \frac{232}{15}\zeta_2^2 \right) \\ & + C_F C_A n_f \left(20 - \frac{1336}{27}\zeta_2 + \frac{200}{9}\zeta_3 + \frac{4}{5}\zeta_2^2 \right) \end{aligned} \tag{2.81}$$

and

$$\beta_1 = \frac{34}{3} C_A^2 - 2 C_F n_f - \frac{10}{3} C_A n_f \,, \tag{2.82}$$

we find, using eq. (2.67),

$$\begin{aligned} H_3 = & \; C_F^3 \left(\frac{29}{2} + 18\zeta_2 + 68\zeta_3 + \frac{288}{5}\zeta_2^2 - 32\zeta_2\zeta_3 - 240\zeta_5 \right) \\ & + C_F^2 C_A \left(\frac{94}{3} - \frac{1235}{3}\zeta_2 + \frac{2296}{3}\zeta_3 - \frac{856}{15}\zeta_2^2 + 16\zeta_2\zeta_3 + 120\zeta_5 \right) \\ & + C_F C_A^2 \left(\frac{16540}{27} + \frac{22286}{27}\zeta_2 - \frac{1544}{3}\zeta_3 - \frac{1592}{15}\zeta_2^2 + 40\zeta_5 \right) \\ & + C_F^2 n_f \left(-\frac{239}{6} + \frac{146}{3}\zeta_2 - \frac{400}{3}\zeta_3 + \frac{208}{15}\zeta_2^2 \right) + C_F n_f^2 \left(\frac{406}{27} + \frac{536}{27}\zeta_2 \right) \\ & + C_F C_A n_f \left(-\frac{5516}{27} - \frac{7216}{27}\zeta_2 + \frac{224}{3}\zeta_3 + \frac{296}{15}\zeta_2^2 \right) \,. \end{aligned} \tag{2.83}$$

Now we have [5]

$$\begin{aligned}B_3 = &C_F^3\left(-\frac{29}{2}-18\zeta_2-68\zeta_3-\frac{288}{5}\zeta_2^2+32\zeta_2\zeta_3+240\zeta_5\right)\\&+C_F^2C_A\left(-46+287\zeta_2-\frac{712}{3}\zeta_3-\frac{272}{5}\zeta_2^2-16\zeta_2\zeta_3-120\zeta_5\right)\\&+C_FC_A^2\left(-\frac{599375}{729}+\frac{32126}{81}\zeta_2+\frac{21032}{27}\zeta_3-\frac{652}{15}\zeta_2^2-\frac{176}{3}\zeta_2\zeta_3-232\zeta_5\right)\\&+C_F^2n_f\left(\frac{5501}{54}-50\zeta_2+\frac{32}{9}\zeta_3\right)+C_Fn_f^2\left(-\frac{8714}{729}+\frac{232}{27}\zeta_2-\frac{32}{27}\zeta_3\right)\\&+C_FC_An_f\left(\frac{160906}{729}-\frac{9920}{81}\zeta_2-\frac{776}{9}\zeta_3+\frac{208}{15}\zeta_2^2\right)\end{aligned}\tag{2.84}$$

and [13]

$$\begin{aligned}G_3 = &C_F^3\left(29+36\zeta_2+136\zeta_3+\frac{576}{5}\zeta_2^2-64\zeta_2\zeta_3-480\zeta_5\right)\\&+C_F^2C_A\left(\frac{232}{3}-\frac{2096}{3}\zeta_2+\frac{3008}{3}\zeta_3-\frac{8}{3}\zeta_2^2+32\zeta_2\zeta_3+240\zeta_5\right)\\&+C_FC_A^2\left(\frac{1045955}{729}+\frac{34732}{81}\zeta_2-\frac{34928}{27}\zeta_3-\frac{188}{3}\zeta_2^2+\frac{176}{3}\zeta_2\zeta_3+272\zeta_5\right)\\&+C_F^2n_f\left(-\frac{3826}{27}+\frac{296}{3}\zeta_2-\frac{1232}{9}\zeta_3+\frac{208}{15}\zeta_2^2\right)+C_Fn_f^2\left(\frac{19676}{729}+\frac{304}{27}\zeta_2+\frac{32}{27}\zeta_3\right)\\&+C_FC_An_f\left(-\frac{309838}{729}-\frac{11728}{81}\zeta_2+\frac{1448}{9}\zeta_3+\frac{88}{15}\zeta_2^2\right),\end{aligned}\tag{2.85}$$

so that we indeed get

$$G_3 = H_3 - B_3\,,\tag{2.86}$$

checking eq. (2.40) for $i=3$. We again note the simple relations between the C_F^3 terms in eqs. (2.81), (2.83), (2.84) and (2.85).

General structure of $H(a_s)$ and $B(a_s)$: we observe that

$$\begin{aligned}C_{\rm DIS}(a_s)+A(a_s)\gamma_E-\beta(a_s)\frac{\partial}{\partial a_s}\ln g_0^{\rm DIS}(1,a_s) &\equiv \beta(a_s)\frac{\partial}{\partial a_s}\Delta_{\rm DIS}(a_s)\\&=-\beta_0\Delta_1^{\rm DIS}a_s^2-(\beta_1\Delta_1^{\rm DIS}+2\beta_0\Delta_2^{\rm DIS})a_s^3+\ldots,\end{aligned}\tag{2.87}$$

where $\Delta_{\rm DIS}(a_s)=\Delta_1^{\rm DIS}a_s+\Delta_2^{\rm DIS}a_s^2+\ldots$, and the beta function factorizes, in the sense that the $\Delta_i^{\rm DIS}$'s are group theory factors polynomials:

$$\begin{aligned}\Delta_1^{\rm DIS} &= C_F(9+4\zeta_2)\\\Delta_2^{\rm DIS} &= C_F\left[\beta_0\left(\frac{457}{24}+19\zeta_2+2\zeta_3\right)\right.\\&\quad\left.+C_F\left(-\frac{7}{8}-\frac{75}{2}\zeta_2+66\zeta_3+\frac{6}{5}\zeta_2^2\right)+C_A\left(\frac{73}{12}+14\zeta_2-54\zeta_3-\frac{71}{5}\zeta_2^2\right)\right].\end{aligned}\tag{2.88}$$

We thus obtain the general structure

$$H(a_s) = B_\delta(a_s) - \beta(a_s)\frac{\partial}{\partial a_s}\Delta_{\rm DIS}(a_s)\ . \tag{2.89}$$

From eqs. (2.5) and (2.89) we further obtain the general expression for B:

$$B(a_s) = B_\delta(a_s) - G(1,a_s) - \beta(a_s)\frac{\partial}{\partial a_s}\Delta_{\rm DIS}(a_s)\,, \tag{2.90}$$

which actually allows to compute B_i (and in particular B_3) given the universal virtual quantities B_i^δ, G_i, and lower order coefficients with $j < i$ contained in $\Delta_{\rm DIS}(a_s)$. We also note that the combination $B_4 - B_4^\delta + G_4$ among $i = 4$ (not yet computed) coefficients can be determined in terms of known $i \leq 3$ coefficients. Eq. (2.90) is a new result of the present approach.

3. Threshold resummation of the physical anomalous dimension (DY case)

In the DY case, the analogue of the resummation formula eq. (2.2) for the short distance coefficient function is

$$\sigma_{\rm DY}(Q^2,N,\mu^2) \sim g_{\rm DY}(Q^2,\mu^2)\ \exp[E_{\rm DY}(Q^2,N,\mu^2)]\,, \tag{3.1}$$

with

$$E_{\rm DY}(Q^2,N,\mu^2) = \int_0^1 dz\ 2\frac{z^{N-1}-1}{1-z}\left[\int_{\mu^2}^{(1-z)^2Q^2}\frac{dk^2}{k^2}A\left(a_s(k^2)\right) + \frac{1}{2}D\left(a_s((1-z)^2Q^2)\right)\right], \tag{3.2}$$

where

$$D(a_s) = \sum_{i=1}^{\infty} D_i a_s^i \tag{3.3}$$

is the standard Sudakov anomalous dimension which controls large angle soft gluon emission in the DY process, whereas

$$g_{\rm DY}(Q^2,\mu^2) = 1 + \sum_{i=1}^{\infty} g_i^{\rm DY}\left(\frac{Q^2}{\mu^2}\right) a_s^i(\mu^2) \tag{3.4}$$

collects the constant terms not included in $E_{\rm DY}$. Taking the logarithmic derivative of eq. (3.1) one gets at large-N

$$\frac{d\ln\sigma_{\rm DY}(Q^2,N,\mu^2)}{d\ln Q^2} \sim \int_0^1 dz\ 2\frac{z^{N-1}-1}{1-z}\mathcal{S}[(1-z)^2Q^2] + K\left(a_s(Q^2)\right)\ , \tag{3.5}$$

where the "Sudakov effective charge"

$$\mathcal{S}(k^2) = A\left(a_s(k^2)\right) + \frac{1}{2}\frac{dD\left(a_s(k^2)\right)}{d\ln k^2} \tag{3.6}$$

and the "leftover" constant terms function (not to be confused with the form factor related K counterterm in eq. (2.28))

$$K\left(a_s(Q^2)\right) = \frac{d\ln g_{\rm DY}(Q^2,\mu^2)}{d\ln Q^2} \tag{3.7}$$

are renormalization group invariant quantities, and $\mathcal{S}$ refers to the "soft" scale $(1-z)^2Q^2$ in eq. (3.5). Changing variables to $k^2=(1-z)^2Q^2$, eq. (3.5) becomes

$$\frac{d\ln\sigma_{\rm DY}(Q^2,N,\mu^2)}{d\ln Q^2} \sim \int_0^{Q^2}\frac{dk^2}{k^2}F_{\rm DY}\left(\frac{k}{Q},N\right)\mathcal{S}(k^2)+K\left(a_s(Q^2)\right)\,, \tag{3.8}$$

with

$$F_{\rm DY}\left(\frac{k}{Q},N\right) = \left(1-\frac{k}{Q}\right)^{N-1}-1\,. \tag{3.9}$$

It was further shown in [10] that eq. (3.8) is equivalent, up to corrections which vanish for $N\to\infty$, to

$$\frac{d\ln\sigma_{\rm DY}(Q^2,N,\mu^2)}{d\ln Q^2} \sim \int_0^{Q^2}\frac{dk^2}{k^2}G_{\rm DY}\left(\frac{Nk}{Q}\right)\mathcal{S}(k^2)+K\left(a_s(Q^2)\right)\,, \tag{3.10}$$

with

$$G_{\rm DY}\left(\frac{Nk}{Q}\right) = \exp\left(-\frac{Nk}{Q}\right)-1\,, \tag{3.11}$$

where $G_{\rm DY}(Nk/Q)$ is obtained by taking the $N\to\infty$ limit of $F_{\rm DY}(k/Q,N)$ with Nk/Q fixed. The analogues of the large-N relations eqs. (2.22), (2.23) and (2.24) are

$$\frac{d\ln\sigma_{\rm DY}(Q^2,N,\mu^2)}{d\ln Q^2} \sim \int_0^{\infty}\frac{dk^2}{k^2}G_{\rm DY}\left(\frac{Nk}{Q}\right)\mathcal{S}(k^2)+\left[K\left(a_s(Q^2)\right)+\int_{Q^2}^{\infty}\frac{dk^2}{k^2}\mathcal{S}(k^2)\right]\,, \tag{3.12}$$

$$\frac{d\ln\sigma_{\rm DY}(Q^2,N,\mu^2)}{d\ln Q^2} \sim \int_0^{\infty}\frac{dk^2}{k^2}\left[G_{\rm DY}\left(\frac{Nk}{Q}\right)+1\right]\mathcal{S}(k^2)+\left[K\left(a_s(Q^2)\right)-\int_0^{Q^2}\frac{dk^2}{k^2}\mathcal{S}(k^2)\right] \tag{3.13}$$

and

$$\frac{d^2\ln\sigma_{\rm DY}(Q^2,N,\mu^2)}{(d\ln Q^2)^2} \sim \int_0^{\infty}\frac{dk^2}{k^2}\dot{G}_{\rm DY}\left(\frac{Nk}{Q}\right)\mathcal{S}(k^2)+\left[\frac{dK}{d\ln Q^2}-\mathcal{S}(Q^2)\right]\,, \tag{3.14}$$

where $\dot{G}_{\rm DY}\equiv -dG_{\rm DY}/d\ln k^2$, whereas the analogue of eq. (2.26) is

$$\frac{d^2\ln|\mathcal{F}(-Q^2)|^2}{(d\ln Q^2)^2} = \frac{dK}{d\ln Q^2}-\mathcal{S}(Q^2)\,, \tag{3.15}$$

where $\mathcal{F}(-Q^2)$ is the time-like quark form factor.

To compute the left-hand side of eq. (3.15), it is convenient to write

$$\ln|\mathcal{F}(-Q^2)|^2 = \ln\left(\mathcal{F}(Q^2)\right)^2+\mathcal{R}\left(a_s(Q^2)\right)\,, \tag{3.16}$$

with

$$\mathcal{R}\left(a_s(Q^2)\right) \equiv \ln\left|\frac{\mathcal{F}(-Q^2)}{\mathcal{F}(Q^2)}\right|^2, \tag{3.17}$$

and we thus get, using eqs. (3.15) and (2.33)

$$\mathcal{S}(Q^2) = A\left(a_s(Q^2)\right) + \beta\left(a_s(Q^2)\right)\frac{\partial}{\partial a_s}\left[-G\left(1, a_s(Q^2)\right) - \beta\left(a_s(Q^2)\right)\frac{\partial \mathcal{R}}{\partial a_s} + K\left(a_s(Q^2)\right)\right], \tag{3.18}$$

which implies, comparing with eq. (3.6)

$$\frac{1}{2}D(a_s) = -G(1, a_s) - \beta(a_s)\frac{\partial \mathcal{R}}{\partial a_s} + K(a_s)\;. \tag{3.19}$$

We thus have to check that

$$G(1, a_s) + \beta(a_s)\frac{\partial \mathcal{R}}{\partial a_s} = K(a_s) - \frac{1}{2}D(a_s)\;. \tag{3.20}$$

Now $K(a_s)$ can be computed from the analogues of eqs. (2.41) and (2.48) which yield:

$$K(a_s) + C_{\rm DY}(a_s) = 2[B_\delta(a_s) - A(a_s)\gamma_E] + \beta(a_s)\frac{\partial}{\partial a_s}\ln g_0^{\rm DY}(1, a_s)\,, \tag{3.21}$$

where $C_{\rm DY}$ collects the large-N constant terms included in the Sudakov integrals in eq. (3.8) or (3.10), and the factor of two on the right-hand side arises because we have two incoming partons. Similarly to eq. (2.59) we thus have

$$K(a_s) = 2B_\delta(a_s) - [C_{\rm DY}(a_s) + 2A(a_s)\gamma_E] + \beta(a_s)\frac{\partial}{\partial a_s}\ln g_0^{\rm DY}(1, a_s)\;. \tag{3.22}$$

Moreover the analogue of eq. (2.58) is:

$$\begin{aligned} C_{\rm DY}(a_s) \;=\; & 2\mathcal{S}_1 c_0 a_s + (-4\beta_0\mathcal{S}_1 c_1 + 2\mathcal{S}_2 c_0)a_s^2 \\ & +[8\beta_0^2\mathcal{S}_1 c_2 - 4(\beta_1\mathcal{S}_1 + 2\beta_0\mathcal{S}_2)c_1 + 2\mathcal{S}_3 c_0]a_s^3 + \dots\,, \end{aligned} \tag{3.23}$$

where $\mathcal{S}_i$ are defined by

$$\begin{aligned} \mathcal{S}(k^2) \;=\; & \mathcal{S}_1 a_s(Q^2) + \left(-\beta_0\mathcal{S}_1\ln\left(\frac{k^2}{Q^2}\right) + \mathcal{S}_2\right)a_s^2(Q^2) \\ & + \left(\beta_0^2\mathcal{S}_1\ln^2\left(\frac{k^2}{Q^2}\right) - (\beta_1\mathcal{S}_1 + 2\beta_0\mathcal{S}_2)\ln\left(\frac{k^2}{Q^2}\right) + \mathcal{S}_3\right)a_s^3(Q^2) + \dots\,. \end{aligned} \tag{3.24}$$

Eq. (3.23) is easily obtained from eq. (2.58) once one notices that

$$J_p(N) \equiv \int_0^{Q^2}\frac{dk^2}{k^2}\left[\exp\left(-\frac{Nk}{Q}\right) - 1\right]\ln^p\left(\frac{k^2}{Q^2}\right) = 2^{p+1}I_p(N)\;. \tag{3.25}$$

Furthermore eq. (3.6) gives

$$\mathcal{S}_1 \;=\; A_1\,, \tag{3.26}$$

$$\mathcal{S}_2 \;=\; A_2 - \beta_0\frac{1}{2}D_1 \tag{3.27}$$

and

$$\mathcal{S}_3 = A_3 - \beta_1 \frac{1}{2} D_1 - 2\beta_0 \frac{1}{2} D_2 \ . \tag{3.28}$$

From eq. (3.23) one can then infer the general structure (similar to eq. (2.87))

$$\begin{aligned} C_{\rm DY}(a_s) + 2A(a_s)\gamma_E - \beta(a_s)\frac{\partial}{\partial a_s} \ln g_0^{\rm DY}(1,a_s) &\equiv \beta(a_s)\frac{\partial}{\partial a_s}\Delta_{\rm DY}(a_s) \\ &= -\beta_0 \Delta_1^{\rm DY} a_s^2 - (\beta_1 \Delta_1^{\rm DY} + 2\beta_0 \Delta_2^{\rm DY}) a_s^3 + \dots \,, \end{aligned} \tag{3.29}$$

where $\Delta_{\rm DY}(a_s) = \Delta_1^{\rm DY} a_s + \Delta_2^{\rm DY} a_s^2 + \ldots$, and the $\Delta_i^{\rm DY}$'s are group theory factors polynomials, which yields the general structure of $K(a_s)$

$$K(a_s) = 2B_\delta(a_s) - \beta(a_s)\frac{\partial}{\partial a_s}\Delta_{\rm DY}(a_s) \ . \tag{3.30}$$

One finds:

(i)

$$\Delta_1^{\rm DY} = C_F(16 - 8\zeta_2) \,, \tag{3.31}$$

where we used [17]

$$D_1 = 0 \tag{3.32}$$

and

$$g_{01}^{\rm DY} = C_F\left(-16 + 8\gamma_E^2 + 16\zeta_2\right) \ . \tag{3.33}$$

(ii)

$$\begin{aligned} \Delta_2^{\rm DY} = C_F \Bigg[& \beta_0 \left(\frac{127}{4} - \frac{56}{3}\zeta_2 + 12\zeta_3 \right) \\ & + C_F\left(\frac{1}{4} - 58\zeta_2 + 60\zeta_3 + \frac{88}{5}\zeta_2^2 \right) + C_A \left(\frac{23}{2} + \frac{8}{3}\zeta_2 - 72\zeta_3 + \frac{12}{5}\zeta_2^2 \right) \Bigg] \,, \end{aligned} \tag{3.34}$$

where we used [17]

$$D_2 = C_F C_A \left(-\frac{1616}{27} + \frac{176}{3}\zeta_2 + 56\zeta_3 \right) + C_F n_f \left(\frac{224}{27} - \frac{32}{3}\zeta_2 \right) \tag{3.35}$$

and

$$\begin{aligned} g_{02}^{\rm DY} = C_F^2 & \left(\frac{511}{4} - 128\gamma_E^2 + 32\gamma_E^4 - 198\zeta_2 + 128\gamma_E^2\zeta_2 - 60\zeta_3 + \frac{552}{5}\zeta_2^2 \right) \\ & + C_F C_A \Bigg(-\frac{1535}{12} + \frac{1616}{27}\gamma_E + \frac{536}{9}\gamma_E^2 + \frac{176}{9}\gamma_E^3 + \frac{376}{3}\zeta_2 \\ & \qquad -16\gamma_E^2\zeta_2 + \frac{604}{9}\zeta_3 - 56\gamma_E\zeta_3 - \frac{92}{5}\zeta_2^2 \Bigg) \\ & + C_F n_f \left(\frac{127}{6} - \frac{224}{27}\gamma_E - \frac{80}{9}\gamma_E^2 - \frac{32}{9}\gamma_E^3 - \frac{64}{3}\zeta_2 + \frac{8}{9}\zeta_3 \right) \ . \end{aligned} \tag{3.36}$$

Eq. (3.30) thus yields the following results in low orders:

- For $i=1$

$$K_1 = 2B_1^\delta \ . \tag{3.37}$$

- For $i=2$

$$K_2 = 2B_2^\delta + C_F\beta_0(16-8\zeta_2) \ . \tag{3.38}$$

- For $i=3$

$$\begin{aligned} K_3 \ &= 2B_3^\delta + C_F\beta_1\left(16-8\zeta_2\right) + C_F\beta_0^2\left(\frac{127}{2} - \frac{112}{3}\zeta_2 + 24\zeta_3\right) \\ &+C_F\beta_0\left[C_F\left(\frac{1}{2}-116\zeta_2+120\zeta_3+\frac{176}{5}\zeta_2^2\right)+C_A\left(23+\frac{16}{3}\zeta_2-144\zeta_3+\frac{24}{5}\zeta_2^2\right)\right] \ . \end{aligned} \tag{3.39}$$

We can now check eq. (3.20) for $i \leq 3$, and thus prove eq. (3.15) to order a_s^4. We first note that from [12, 13] one gets

$$\mathcal{R}(a_s) = 3\zeta_2 A_1 a_s + 3\zeta_2(\beta_0 G_1 + A_2)a_s^2 + \ldots \equiv r_1 a_s + r_2 a_s^2 + \ldots \ , \tag{3.40}$$

which yields

$$\begin{aligned} r_1 \ &= 12C_F\zeta_2 \\ r_2 \ &= 12C_F\zeta_2\left[\left(\frac{233}{18} - 2\zeta_2\right)C_A - \frac{19}{9}n_f\right] \ . \end{aligned} \tag{3.41}$$

Since $D_1 = 0$, the order a_s contribution to the right-hand side of eq. (3.20) reduces to K_1, whereas $\mathcal{R}$ does not contribute at this order to the left-hand side, which reduces to G_1. Thus one has to check that

$$G_1 = K_1 \,, \tag{3.42}$$

which is indeed satisfied (see eq. (3.37) and the relevant expressions in section 2). Next we get from eq. (3.38)

$$K_2 = C_F C_A\left(\frac{193}{3} - 24\zeta_3\right) - C_F n_f\frac{34}{3} + C_F^2\left(3 - 24\zeta_2 + 48\zeta_3\right) \ , \tag{3.43}$$

whereas the order a_s^2 contribution to the left-hand side of eq. (3.20) is $G_2 - \beta_0 r_1$. So we should check whether

$$G_2 = K_2 - \frac{1}{2}D_2 + \beta_0 r_1 \,, \tag{3.44}$$

which is also satisfied.

Finally we have from eq. (3.39)

$$\begin{aligned} K_3 \ &= C_F^3\left(29 + 36\zeta_2 + 136\zeta_3 + \frac{576}{5}\zeta_2^2 - 64\zeta_2\zeta_3 - 480\zeta_5\right) \\ &+C_F^2 C_A\left(\frac{232}{3} - \frac{2096}{3}\zeta_2 + \frac{3008}{3}\zeta_3 - \frac{8}{3}\zeta_2^2 + 32\zeta_2\zeta_3 + 240\zeta_5\right) \\ &+C_F C_A^2\left(\frac{3082}{3} - 240\zeta_2 - \frac{4952}{9}\zeta_3 + \frac{68}{5}\zeta_2^2 + 80\zeta_5\right) \\ &+C_F^2 n_f\left(-\frac{235}{3} + \frac{320}{3}\zeta_2 - \frac{512}{3}\zeta_3 + \frac{112}{15}\zeta_2^2\right) + C_F n_f^2\left(\frac{220}{9} - \frac{32}{3}\zeta_2 + \frac{64}{9}\zeta_3\right) \\ &+C_F C_A n_f\left(-\frac{3052}{9} + \frac{320}{3}\zeta_2 + \frac{208}{9}\zeta_3 - \frac{8}{5}\zeta_2^2\right) \end{aligned} \tag{3.45}$$

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and [17, 18]

$$D_3 = C_F C_A^2 \left(-\frac{594058}{729} + \frac{98224}{81}\zeta_2 + \frac{40144}{27}\zeta_3 - \frac{2992}{15}\zeta_2^2 - \frac{352}{3}\zeta_2\zeta_3 - 384\zeta_5\right)$$
$$+C_F^2 n_f \left(\frac{3422}{27} - 32\zeta_2 - \frac{608}{9}\zeta_3 - \frac{64}{5}\zeta_2^2\right) + C_F n_f^2 \left(-\frac{3712}{729} + \frac{640}{27}\zeta_2 + \frac{320}{27}\zeta_3\right)$$
$$+C_F C_A n_f \left(\frac{125252}{729} - \frac{29392}{81}\zeta_2 - \frac{2480}{9}\zeta_3 + \frac{736}{15}\zeta_2^2\right) . \tag{3.46}$$

Now the order a_s^3 contribution to the left-hand side of eq. (3.20) is $G_3 - (\beta_1 r_1 + 2\beta_0 r_2)$. So we should check whether

$$G_3 = K_3 - \frac{1}{2}D_3 + \beta_1 r_1 + 2\beta_0 r_2 \, , \tag{3.47}$$

which is again satisfied.

General structure of $D(a_s)$: from eqs. (3.20) and (3.30) we further obtain the following general expression for D (the analogue of eq. (2.90)):

$$\frac{1}{2}D(a_s) = 2B_\delta(a_s) - G(1, a_s) - \beta(a_s)\frac{\partial \mathcal{R}}{\partial a_s} - \beta(a_s)\frac{\partial}{\partial a_s}\Delta_{\rm DY}(a_s) \, , \tag{3.48}$$

which allows to compute D_i given the universal virtual quantities B_i^δ, G_i, and lower order $j \leq i$ coefficients. Eq. (3.48) is actually closely related to results given in [17] and [18]. To make contact with the "universal" quantities $f_q(a_s)$ appearing in [17] and defined in [19, 20], we can put

$$G(1, a_s) \equiv \tilde{G}(a_s) + \Delta G(a_s) \, , \tag{3.49}$$

with $\tilde{G}$ as defined in [18, 21], such that

$$f_q(a_s) \equiv \tilde{G}(a_s) - 2B_\delta(a_s) \; . \tag{3.50}$$

We note that $\Delta G(a_s)$ is also proportional to the beta function, i.e. has the structure

$$\Delta G(a_s) = \beta(a_s)\frac{\partial \kappa}{\partial a_s} \, , \tag{3.51}$$

where $\kappa(a_s)$ is a power series with polynomial dependence on the group theory factors. Then eq. (3.48) becomes

$$\frac{1}{2}D(a_s) = -f_q(a_s) - \beta(a_s)\frac{\partial}{\partial a_s}\left[\mathcal{R}(a_s) + \kappa(a_s) + \Delta_{\rm DY}(a_s)\right] \, , \tag{3.52}$$

which should be equivalent to eq. (4.4) in [18], and reproduces eq. (36) in [17].

As a last comment, we note that subtracting eq. (2.90) from eq. (3.48) we obtain

$$\frac{1}{2}D(a_s) - B(a_s) - B_\delta(a_s) = -\beta(a_s)\frac{\partial}{\partial a_s}\left[\mathcal{R}(a_s) + \Delta_{\rm DY}(a_s) - \Delta_{\rm DIS}(a_s)\right] \, , \tag{3.53}$$

which yields the relations [5]

$$\begin{aligned}\frac{1}{2}D_1 - B_1 - B_1^\delta &= 0,\\ \frac{1}{2}D_2 - B_2 - B_2^\delta &= 7C_F\beta_0\,,\end{aligned} \tag{3.54}$$

as well as the new relation

$$\begin{aligned}\frac{1}{2}D_3 - B_3 - B_3^\delta = 7C_F\beta_1 + \beta_0\Bigg[&C_AC_F\left(\frac{65}{6}+\frac{28}{3}\zeta_2 - 36\zeta_3 - \frac{75}{5}\zeta_2^2\right)\\ &+\beta_0 C_F\left(\frac{305}{12}+\frac{2}{3}\zeta_2+20\zeta_3\right)+C_F^2\left(\frac{9}{4}-41\zeta_2-12\zeta_3+\frac{164}{5}\zeta_2^2\right)\Bigg]\end{aligned} \tag{3.55}$$

allowing to compute D_3 given B_3, B_3^δ, and information obtained from $i \leq 2$ coefficients. We also note that the combination $\frac{1}{2}D_4 - B_4 - B_4^\delta$ can be determined in terms of known $i \leq 3$ coefficients.

4. The variety of resummation procedures

4.1 DIS case

It was observed in [10, 11] that the separation between the constant terms contained in the Sudakov integrals on the right-hand side of eq. (2.13) or (2.15) and the "leftover" constant terms contained in $H(a_s)$ is arbitrary, yielding a variety of Sudakov resummation procedures, different choices leading to a different "Sudakov distribution function" $G_{\rm DIS}^{\rm new}(Nk^2/Q^2)$ and "Sudakov effective coupling" $\mathcal{J}_{\rm new}(k^2)$, as well as to a different function $H_{\rm new}(a_s)$, namely we have the alternative large-N representations (up to terms which vanish order by order in perturbation theory for $N \to \infty$)

$$\frac{d\ln F_2(Q^2,N)}{d\ln Q^2} \sim S_{\rm DIS}^{\rm new}(Q^2,N) + H_{\rm new}\left(a_s(Q^2)\right)\,, \tag{4.1}$$

with

$$S_{\rm DIS}^{\rm new}(Q^2,N) = \int_0^{Q^2}\frac{dk^2}{k^2}G_{\rm DIS}^{\rm new}\left(\frac{Nk^2}{Q^2}\right)\mathcal{J}_{\rm new}(k^2)\,. \tag{4.2}$$

As in eq. (2.18), we can extend to infinity the upper limit of integration to obtain

$$\begin{aligned}S_{\rm DIS}^{\rm new}(Q^2,N) &\sim \int_0^\infty\frac{dk^2}{k^2}G_{\rm DIS}^{\rm new}\left(\frac{Nk^2}{Q^2}\right)\mathcal{J}_{\rm new}(k^2) - G_{\rm DIS}^{\rm new}(\infty)\int_{Q^2}^\infty\frac{dk^2}{k^2}\mathcal{J}_{\rm new}(k^2)\\ &= \int_0^\infty\frac{dk^2}{k^2}G_{\rm DIS}^{\rm new}\left(\frac{Nk^2}{Q^2}\right)\mathcal{J}_{\rm new}(k^2) + \int_{Q^2}^\infty\frac{dk^2}{k^2}\mathcal{J}_{\rm new}(k^2)\,,\end{aligned} \tag{4.3}$$

where in the second line, we used that $G_{\rm DIS}^{\rm new}(\infty) = -1$ for all resummation procedures, corresponding to the virtual contribution in the Sudakov integral (which determines the leading logs of N). Thus we get

$$\int_0^\infty\frac{dk^2}{k^2}G_{\rm DIS}^{\rm new}\left(\frac{Nk^2}{Q^2}\right)\mathcal{J}_{\rm new}(k^2) + \int_{Q^2}^\infty\frac{dk^2}{k^2}\mathcal{J}_{\rm new}(k^2) = \sum_{i=1}^\infty \gamma_i^{\rm new}(N)\ a_s^i(Q^2)\,, \tag{4.4}$$

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with ($L \equiv \ln N$)

$$\begin{aligned} \gamma_1^{\rm new}(N) &= \gamma_{11} L + \gamma_{10}^{\rm new} \\ \gamma_2^{\rm new}(N) &= \gamma_{22} L^2 + \gamma_{21} L + \gamma_{20}^{\rm new} \\ \gamma_3^{\rm new}(N) &= \gamma_{33} L^3 + \gamma_{32} L^2 + \gamma_{31} L + \gamma_{30}^{\rm new} \\ & \textit{etc.}\ , \end{aligned} \tag{4.5}$$

where only the *non-logarithmic* $\gamma_{i0}^{\rm new}$ terms do depend upon the resummation procedure.

Alternatively, as in the standard case, one may remove the virtual contribution from the Sudakov integral (so that it contains only real gluon emission contributions), and merge it together with the "leftover" constant terms, which yields the equivalent result, in terms of two separately IR divergent (but UV finite) integrals

$$S_{\rm DIS}^{\rm new}(Q^2,N) \sim \int_0^\infty \frac{dk^2}{k^2}\left[G_{\rm DIS}^{\rm new}\left(\frac{Nk^2}{Q^2}\right)+1\right]\mathcal{J}_{\rm new}(k^2) - \int_0^{Q^2}\frac{dk^2}{k^2}\mathcal{J}_{\rm new}(k^2)\ . \tag{4.6}$$

Using eq. (4.3) into eq. (4.1), we thus end up with the large-N expression

$$\frac{d\ln F_2(Q^2,N)}{d\ln Q^2} \sim \int_0^\infty \frac{dk^2}{k^2} G_{\rm DIS}^{\rm new}\left(\frac{Nk^2}{Q^2}\right)\mathcal{J}_{\rm new}(k^2) + \left[H_{\rm new}\left(a_s(Q^2)\right) + \int_{Q^2}^\infty \frac{dk^2}{k^2}\mathcal{J}_{\rm new}(k^2)\right]. \tag{4.7}$$

If instead one uses eq. (4.6) into eq. (4.1) one gets the equivalent form

$$\frac{d\ln F_2(Q^2,N)}{d\ln Q^2} \sim \int_0^\infty \frac{dk^2}{k^2}\left[G_{\rm DIS}^{\rm new}\left(\frac{Nk^2}{Q^2}\right)\right]\mathcal{J}_{\rm new}(k^2) + \left[H_{\rm new}\left(a_s(Q^2)\right) - \int_0^{Q^2}\frac{dk^2}{k^2}\mathcal{J}_{\rm new}(k^2)\right]. \tag{4.8}$$

Again we observe [3] that the UV (respectively IR) divergences present in the individual integrals in eq. (4.7) (respectively eq. (4.8)) disappear after taking one more derivative (which eliminates the virtual contribution inside the Sudakov integral), namely

$$\frac{d^2\ln F_2(Q^2,N)}{(d\ln Q^2)^2} \sim \int_0^\infty \frac{dk^2}{k^2}\dot{G}_{\rm DIS}^{\rm new}\left(\frac{Nk^2}{Q^2}\right)\mathcal{J}_{\rm new}(k^2) + \left[\frac{dH_{\rm new}}{d\ln Q^2} - \mathcal{J}_{\rm new}(Q^2)\right], \tag{4.9}$$

where $\dot{G}_{\rm DIS}^{\rm new} \equiv -dG_{\rm DIS}^{\rm new}/d\ln k^2$, and the integral in eq. (4.9)

$$S'_{\rm DIS}\left(\frac{Q^2}{N}\right) \equiv \int_0^\infty \frac{dk^2}{k^2}\dot{G}_{\rm DIS}^{\rm new}\left(\frac{Nk^2}{Q^2}\right)\mathcal{J}_{\rm new}(k^2) \tag{4.10}$$

is finite, and consequently uniquely determined, without need for a "*new*" subscript anymore. The point is that $S'_{\rm DIS}(Q^2/N)$ being UV (and IR) convergent, all the large-N logarithmic terms (which are unambiguously fixed) are now determined by the constant terms contained in the integral, which cannot be fixed arbitrarily anymore. This observation implies in turn that the combination $dH_{\rm new}/d\ln Q^2 - \mathcal{J}_{\rm new}(Q^2)$, which represents the "leftover" constant terms not included in $S'_{\rm DIS}(Q^2/N)$, is also uniquely fixed. Consequently [3] the conjecture eq. (2.26) has an analogue for *all* resummation procedures, namely

$$\frac{d^2\ln\left(\mathcal{F}(Q^2)\right)^2}{(d\ln Q^2)^2} = \frac{dH_{\rm new}}{d\ln Q^2} - \mathcal{J}_{\rm new}(Q^2)\ . \tag{4.11}$$

The same unicity statements are actually valid in a more formal[4] sense (since they are UV or

[4] A rigorous definition can be given in term of Borel transforms, which are however singular at the origin.

IR divergent quantities) for the integrals (which could both be referred to as $S_{\rm DIS}(Q^2/N)$) appearing on the first line of eq. (4.7) or (4.8), as well as for the combination of constant terms appearing on the second line of these equations. In particular, the second line of eq. (4.8) can be formally identified, as in the standard procedure (section 2) to the first logarithmic derivative of the square of the quark form factor, an IR divergent[5] quantity:

$$\frac{d\ln\left(\mathcal{F}(Q^2)\right)^2}{d\ln Q^2} = H_{\rm new}\left(a_s(Q^2)\right) - \int_0^{Q^2}\frac{dk^2}{k^2}\mathcal{J}_{\rm new}(k^2)\ . \tag{4.12}$$

An application of this relation is given in section 5 to the "Minkowskian" resummation formalism.

Finally, following steps analoguous to those which lead to eq. (2.36), one can show that eq. (4.11) is equivalent to

$$\begin{aligned}\mathcal{J}_{\rm new}(Q^2) &= A\left(a_s(Q^2)\right) + \beta\left(a_s(Q^2)\right)\frac{\partial}{\partial a_s}\left[-G\left(1,a_s(Q^2)\right) + H_{\rm new}\left(a_s(Q^2)\right)\right]\\ &\equiv A\left(a_s(Q^2)\right) + \frac{dB_{\rm new}\left(a_s(Q^2)\right)}{d\ln Q^2}\ ,\end{aligned} \tag{4.13}$$

which implies the new Sudakov anomalous dimension $B_{\rm new}$ should be given by

$$B_{\rm new}(a_s) = -G(1,a_s) + H_{\rm new}(a_s)\ . \tag{4.14}$$

Eq. (4.13) also shows that the Sudakov effective coupling $\mathcal{J}_{\rm new}$ differs from the cusp anomalous dimension by a term proportional to the beta function in *all*[6] resummation procedures.

4.2 DY case

Similar generalizations apply in the DY case. The generalizations of the large-N relations eqs. (3.12), (3.13) and (3.14) are

$$\begin{aligned}\frac{d\ln\sigma_{\rm DY}(Q^2,N,\mu^2)}{d\ln Q^2} &\sim \int_0^\infty\frac{dk^2}{k^2}G_{\rm DY}^{\rm new}\left(\frac{Nk}{Q}\right)\mathcal{S}_{\rm new}(k^2)\\ &\quad+\left[K_{\rm new}\left(a_s(Q^2)\right) + \int_{Q^2}^\infty\frac{dk^2}{k^2}\mathcal{S}_{\rm new}(k^2)\right]\ ,\end{aligned} \tag{4.15}$$

$$\begin{aligned}\frac{d\ln\sigma_{\rm DY}(Q^2,N,\mu^2)}{d\ln Q^2} &\sim \int_0^\infty\frac{dk^2}{k^2}\left[G_{\rm DY}^{\rm new}\left(\frac{Nk}{Q}\right)+1\right]\mathcal{S}_{\rm new}(k^2)\\ &\quad+\left[K_{\rm new}\left(a_s(Q^2)\right) - \int_0^{Q^2}\frac{dk^2}{k^2}\mathcal{S}_{\rm new}(k^2)\right]\end{aligned} \tag{4.16}$$

and

$$\frac{d^2\ln\sigma_{\rm DY}(Q^2,N,\mu^2)}{(d\ln Q^2)^2} \sim \int_0^\infty\frac{dk^2}{k^2}\dot{G}_{\rm DY}^{\rm new}\left(\frac{Nk}{Q}\right)\mathcal{S}_{\rm new}(k^2) + \left[\frac{dK_{\rm new}}{d\ln Q^2} - \mathcal{S}_{\rm new}(Q^2)\right]\ , \tag{4.17}$$

[5]Eq. (4.12) may still make sense at the *non-perturbative* level, if one assumes that the Sudakov effective charge $\mathcal{J}_{\rm new}(k^2)$ vanishes for $k^2 \to 0$. This vanishing also occurs [22, 23] for $D > 4$ in the dimensional regularization framework.

[6]We note however that the general structures eqs. (2.89) and (2.90) hold usually *only* for the standard procedure, since they rely on the specific relations eqs. (2.55), (2.56) and (2.57).

where $\dot{G}_{\rm DY}^{\rm new} \equiv -dG_{\rm DY}^{\rm new}/d\ln k^2$, whereas the conjecture eq. (3.15) implies

$$\frac{d^2 \ln|\mathcal{F}(-Q^2)|^2}{(d\ln Q^2)^2} = \frac{dK_{\rm new}}{d\ln Q^2} - \mathcal{S}_{\rm new}(Q^2)\ . \tag{4.18}$$

Eq. (4.18) is equivalent to the statement that, for any resummation procedure

$$\begin{aligned}\mathcal{S}_{\rm new}(Q^2) &= A\left(a_s(Q^2)\right) + \beta\left(a_s(Q^2)\right)\frac{\partial}{\partial a_s}\Big[-G\left(1,a_s(Q^2)\right) - \beta\left(a_s(Q^2)\right)\frac{\partial \mathcal{R}}{\partial a_s}\\ &\qquad\qquad + K_{\rm new}\left(a_s(Q^2)\right)\Big]\\ &\equiv A\left(a_s(Q^2)\right) + \frac{1}{2}\frac{dD_{\rm new}\left(a_s(Q^2)\right)}{d\ln Q^2}\,,\end{aligned} \tag{4.19}$$

with[7]

$$\frac{1}{2}D_{\rm new}(a_s) = -G(1,a_s) - \beta(a_s)\frac{\partial \mathcal{R}}{\partial a_s} + K_{\rm new}(a_s)\ . \tag{4.20}$$

5. All-order check at large-n_f: connection with the dispersive approach

5.1 DIS case

At large-n_f ("large-β_0" limit) and finite N, the following dispersive representation holds [24, 25]

$$\left.\frac{d\ln F_2(Q^2,N)}{d\ln Q^2}\right|_{\text{large-}\beta_0} = \int_0^\infty \frac{d\lambda^2}{\lambda^2}\ddot{\mathcal{F}}_{\rm DIS}\left(\frac{\lambda^2}{Q^2},N\right)A^V_{\rm Mink}(\lambda^2)\,, \tag{5.1}$$

where

$$\frac{1}{4C_F}A^V_{\rm Mink}(\lambda^2) = \frac{1}{\beta_0}\left[\frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{1}{\pi}\ln\left(\frac{\lambda^2}{\Lambda_V^2}\right)\right)\right] \tag{5.2}$$

is the time-like (integrated) discontinuity of the *Euclidean* one-loop coupling (the "V-scheme" coupling) associated to the dressed gluon propagator

$$\frac{1}{4C_F}A^V_{\rm Eucl}(k^2) = \frac{1}{\beta_0\ln\left(\frac{k^2}{\Lambda_V^2}\right)}\,, \tag{5.3}$$

(Λ_V is the V-scheme scale parameter). Eq. (5.1) represents the "single dressed gluon" exchange contribution, i.e. the infinite sum of diagrams with a single gluon exchange, dressed with an arbitary number of (renormalized) quark loops. Let us now take the large-N limit of eq. (5.1). We shall use the following two properties of the Mellin space characteristic function

$$\mathcal{F}_{\rm DIS}(\epsilon,N) \equiv \int_0^1 dx\ x^{N-1}\tilde{\mathcal{F}}_{\rm DIS}(\epsilon,x)\,, \tag{5.4}$$

where $\epsilon \equiv \lambda^2/Q^2$, and $\tilde{\mathcal{F}}_{\rm DIS}(\epsilon,x)$ is the momentum space characteristic function:

[7] Again (footnote 6), the general structure eq. (3.48) is usually valid only for the standard resummation procedure.

(i) For $N \to \infty$ with $\epsilon_j \equiv N\epsilon$ fixed, we have the scaling property [3, 10, 11] (see appendix B)

$$\ddot{\mathcal{F}}_{\rm DIS}(\epsilon, N) \sim \ddot{\mathcal{G}}_{\rm DIS}(\epsilon_j) \ . \tag{5.5}$$

(ii) For $N \to \infty$ with ϵ fixed we have

$$\ddot{\mathcal{F}}_{\rm DIS}(\epsilon, N) \sim \ddot{\mathcal{V}}_s(\epsilon) \, , \tag{5.6}$$

where[8] [24]

$$\mathcal{V}_s(\epsilon) = -\int_0^1 dz \frac{(1-z)^2}{z-\epsilon} \ln \frac{z}{\epsilon} \tag{5.7}$$

is the virtual contribution to the characteristic function.

Eq. (5.6) follows from the expression [24] for the momentum space characteristic function

$$\tilde{\mathcal{F}}_{\rm DIS}(\epsilon, x) = \tilde{\mathcal{F}}^{(r)}_{\rm DIS}(\epsilon, x)\Theta(1 - x - \epsilon x) + \mathcal{V}_s(\epsilon)\delta(1-x) \, , \tag{5.8}$$

(where $\tilde{\mathcal{F}}^{(r)}_{\rm DIS}(\epsilon, x)$ is the real contribution) which gives in Mellin space

$$\mathcal{F}_{\rm DIS}(\epsilon, N) = \int_0^{\frac{1}{1+\epsilon}} dx \ x^{N-1} \tilde{\mathcal{F}}^{(r)}_{\rm DIS}(\epsilon, x) + \mathcal{V}_s(\epsilon) \ . \tag{5.9}$$

To derive the large-N limit of eq. (5.1), we first take the limit $N \to \infty$ with $N\epsilon$ fixed inside the dispersive integral, getting

$$\left.\frac{d \ln F_2(Q^2, N)}{d \ln Q^2}\right|_{\text{large-}\beta_0} \sim \int_0^\infty \frac{d\lambda^2}{\lambda^2} \ddot{\mathcal{G}}_{\rm DIS}(N\epsilon) A^V_{\rm Mink}(\lambda^2) \, , \tag{5.10}$$

where the right-hand side is however UV divergent, since $\ddot{\mathcal{G}}_{\rm DIS}(\infty) = -1$ (corresponding to the virtual contribution). To introduce the required UV subtraction, we write eq. (5.1) identically as

$$\begin{aligned} \left.\frac{d \ln F_2(Q^2, N)}{d \ln Q^2}\right|_{\text{large-}\beta_0} &= \int_0^\infty \frac{d\lambda^2}{\lambda^2} \ddot{\mathcal{G}}_{\rm DIS}(N\epsilon) A^V_{\rm Mink}(\lambda^2) \\ &\quad + \int_0^\infty \frac{d\lambda^2}{\lambda^2} \left[\ddot{\mathcal{F}}_{\rm DIS}(\epsilon, N) - \ddot{\mathcal{G}}_{\rm DIS}(N\epsilon)\right] A^V_{\rm Mink}(\lambda^2) \, , \end{aligned} \tag{5.11}$$

and take in a second step the limit $N \to \infty$ (with ϵ fixed!) inside the second integral, thus getting[9]

$$\left.\frac{d \ln F_2(Q^2, N)}{d \ln Q^2}\right|_{\text{large-}\beta_0} \sim \int_0^\infty \frac{d\lambda^2}{\lambda^2} \ddot{\mathcal{G}}_{\rm DIS}(N\epsilon) A^V_{\rm Mink}(\lambda^2) + \int_0^\infty \frac{d\lambda^2}{\lambda^2} \left[\ddot{\mathcal{V}}_s(\epsilon) + 1\right] A^V_{\rm Mink}(\lambda^2) \ . \tag{5.12}$$

[8]Our normalization of $\mathcal{F}_{\rm DIS}$ and of $\mathcal{V}_s$ is half that in [24].

[9]Eq. (5.12) is at the basis of the dispersive approach [3, 4] to Sudakov resummation.

The second integral in eq. (5.12) now appears as an N-independent subtraction term, which regulates the UV divergence of the first integral. It is remarkable, on the other hand, that both integrals are IR convergent. Indeed, one finds for $\epsilon \to 0$

$$\ddot{\mathcal{V}}_s(\epsilon) + 1 \sim \epsilon \ln^2 \epsilon \,, \tag{5.13}$$

while, for $\epsilon \to \infty$, $\ddot{\mathcal{V}}_s(\epsilon) = \mathcal{O}(\ln \epsilon/\epsilon)$. Comparing with eq. (4.7) shows that eq. (5.12) is nothing but a peculiar case of eq. (4.7) (at large-β_0) with "*new*"="*Mink*", provided one makes the identifications $G_{\rm DIS}^{\rm Mink}(\epsilon_j) \equiv \ddot{\mathcal{G}}_{\rm DIS}(\epsilon_j)$, $\mathcal{J}_{\rm Mink}(k^2)|_{\text{large-}\beta_0} \equiv A_{\rm Mink}^V(k^2)$, and identifies the N-independent subtraction term on the second line of eq. (4.7) as

$$H_{\rm Mink}\left(a_s(Q^2)\right)\big|_{\text{large-}\beta_0} + \int_{Q^2}^{\infty} \frac{dk^2}{k^2} A_{\rm Mink}^V(k^2) \equiv \int_0^{\infty} \frac{d\lambda^2}{\lambda^2}\left[\ddot{\mathcal{V}}_s(\epsilon) + 1\right] A_{\rm Mink}^V(\lambda^2) \,. \tag{5.14}$$

Eq. (5.14) can be rewritten in terms of UV (and IR) convergent integrals as

$$H_{\rm Mink}\left(a_s(Q^2)\right)\big|_{\text{large-}\beta_0} = \int_0^{Q^2} \frac{d\lambda^2}{\lambda^2}\left[\ddot{\mathcal{V}}_s(\epsilon)+1\right] A_{\rm Mink}^V(\lambda^2) + \int_{Q^2}^{\infty} \frac{d\lambda^2}{\lambda^2} \ddot{\mathcal{V}}_s(\epsilon) A_{\rm Mink}^V(\lambda^2) \,, \tag{5.15}$$

which gives a dispersive representation of $H_{\rm Mink}$ at large-β_0 [10]. On the other hand, both integrals in eq. (5.14) are UV divergent, but this divergence can be disposed of by taking one derivative with respect to $\ln Q^2$, thus getting

$$\left.\frac{dH_{\rm Mink}\left(a_s(Q^2)\right)}{d\ln Q^2}\right|_{\text{large-}\beta_0} - A_{\rm Mink}^V(Q^2) = -\int_0^{\infty} \frac{d\lambda^2}{\lambda^2} \frac{d^3\mathcal{V}_s(\epsilon)}{(d\ln\epsilon)^3} A_{\rm Mink}^V(\lambda^2) \,. \tag{5.16}$$

Since, according to our conjecture (eq. (4.11)), the left-hand side of eq. (5.16) should be equal to $\left.\frac{d^2 \ln\left(\mathcal{F}(Q^2)\right)^2}{(d\ln Q^2)^2}\right|_{\text{large-}\beta_0}$, we only have to check that

$$\left.\frac{d^2 \ln\left(\mathcal{F}(Q^2)\right)^2}{(d\ln Q^2)^2}\right|_{\text{large-}\beta_0} = -\int_0^{\infty} \frac{d\lambda^2}{\lambda^2} \frac{d^3\mathcal{V}_s(\epsilon)}{(d\ln\epsilon)^3} A_{\rm Mink}^V(\lambda^2) \tag{5.17}$$

is the correct dispersive representation of the second logarithmic derivative of the quark form factor in the large-β_0 limit.

Paralleling the discussion in sections 2 and 4, we note that one can also write the right-hand side of eq. (5.12) as the sum of two UV convergent, but IR divergent integrals, by removing the virtual contribution (-1) from the first, N-dependent, integral

$$\left.\frac{d\ln F_2(Q^2,N)}{d\ln Q^2}\right|_{\text{large-}\beta_0} \sim \int_0^{\infty} \frac{d\lambda^2}{\lambda^2}\left[\ddot{\mathcal{G}}_{\rm DIS}(N\epsilon) + 1\right] A_{\rm Mink}^V(\lambda^2) + \int_0^{\infty} \frac{d\lambda^2}{\lambda^2} \ddot{\mathcal{V}}_s(\epsilon) A_{\rm Mink}^V(\lambda^2) \,. \tag{5.18}$$

Eq. (5.18) should be compared to eq. (4.8), leading to the identification

$$H_{\rm Mink}\left(a_s(Q^2)\right)\big|_{\text{large-}\beta_0} - \int_0^{Q^2} \frac{dk^2}{k^2} A_{\rm Mink}^V(k^2) \equiv \int_0^{\infty} \frac{d\lambda^2}{\lambda^2} \ddot{\mathcal{V}}_s(\epsilon) A_{\rm Mink}^V(\lambda^2) \,. \tag{5.19}$$

Comparaison with eq. (4.12) also suggests that formally one can also identify the IR divergent integral on the right-hand side of eq. (5.19) as

$$\left.\frac{d\ln\left(\mathcal{F}(Q^2)\right)^2}{d\ln Q^2}\right|_{\text{large-}\beta_0} = \int_0^\infty \frac{d\lambda^2}{\lambda^2}\ddot{\mathcal{V}}_s(\epsilon)A^V_{\text{Mink}}(\lambda^2)\,, \tag{5.20}$$

a purely virtual contribution, while the first integral on the right-hand side of eq. (5.18) can now be interpreted as containing only real gluon emission contributions.

To check eq. (5.20) (and hence eq. (5.17)), we first observe that the first logarithmic derivative $\frac{1}{\mathcal{F}}\frac{d\mathcal{F}}{d\ln Q^2}$ of the quark form factor coincides in the large-β_0 limit with the ordinary derivative $\frac{d\mathcal{F}}{d\ln Q^2}$. Indeed, since $\mathcal{F} = 1+\mathcal{O}(a_s)$, it is clear that disconnected diagrams coming from the expansion of the denominator $1/\mathcal{F}$ in the logarithmic derivative are subdominant at large-n_f. Thus the problem reduces to find the dispersive representation of $\frac{d\mathcal{F}}{d\ln Q^2}$ in the large-β_0 limit. According to [24, 25], this amounts to the calculation of the characteristic function $\phi(\lambda^2/Q^2)$ of the quark form factor, namely of $\mathcal{F}_1(Q^2) = \phi(\lambda^2/Q^2)\ a_s$, the one loop radiative correction to the on-shell massless quark form factor computed with a finite gluon mass λ. This quantity is expected to be UV divergent, but the divergence should disappear after taking one derivative, namely $\dot{\phi}(\lambda^2/Q^2)$ (which is the characteristic function associated to the derivative of the form factor) should be finite (with $\dot{\phi}(\lambda^2/Q^2) = \mathcal{O}(\ln(\lambda^2))$ for $\lambda^2 \to 0$), yielding the dispersive representation

$$\left.\frac{d\mathcal{F}(Q^2)}{d\ln Q^2}\right|_{\text{large-}\beta_0} = \int_0^\infty \frac{d\lambda^2}{\lambda^2}\ddot{\phi}\left(\frac{\lambda^2}{Q^2}\right)\frac{A^V_{\text{Mink}}(\lambda^2)}{4C_F}\,. \tag{5.21}$$

We note that the integral on the right-hand side of eq. (5.21) should be UV convergent, but IR divergent, since we expect $\ddot{\phi}(\infty) = 0$, but $\ddot{\phi}(0) = \mathcal{O}(1)$. Comparing eq. (5.21) with eq. (5.20) then suggests[10] that

$$\ddot{\phi}\left(\frac{\lambda^2}{Q^2}\right) = 4C_F\frac{1}{2}\ddot{\mathcal{V}}_s(\epsilon) = 2C_F\ddot{\mathcal{V}}_s(\epsilon)\,, \tag{5.22}$$

and also that $\dot{\phi}(\lambda^2/Q^2) = 2C_F\dot{\mathcal{V}}_s(\epsilon)$. Moreover $2C_F\mathcal{V}_s(\epsilon)$, which vanishes at $\epsilon = \infty$, should coincide with the renormalized version $\phi_R(\lambda^2/Q^2)$ of $\phi(\lambda^2/Q^2)$

$$2C_F\mathcal{V}_s(\epsilon) = \phi_R\left(\frac{\lambda^2}{Q^2}\right) \equiv \phi\left(\frac{\lambda^2}{Q^2}\right) - \phi(\infty)\,, \tag{5.23}$$

with the normalization condition $\phi_R(\lambda^2/Q^2) = 0$ at $Q^2 = 0$ (i.e. $\phi_R(\infty) = 0$). These statements are checked in appendix C. We also note that $d\ln\left(\mathcal{F}(Q^2)\right)^2/d\ln Q^2$ has a status similar to that of an infrared and collinear singular quantity, such as $F_2(Q^2,N)$.

A partial check of eq. (5.17) to order a_s^3 can also be performed by comparing the right-hand side expanded to this order with the result of existing order a_s^3 calculations of the quark form factor. This comparaison can be conveniently performed using the Borel transform technique. Indeed in Borel space eq. (5.17) becomes (in the $\overline{MS}$ scheme)

$$B\left[\left.\frac{d^2\ln\left(\mathcal{F}(Q^2)\right)^2}{(d\ln Q^2)^2}\right|_{\text{large-}\beta_0}\right](u) = -4C_F\exp(5u/3)\frac{\sin\pi u}{\pi u}\Gamma_{\text{SDG}}(u)\,, \tag{5.24}$$

[10]The factor $\frac{1}{2}$ in eq. (5.22) arises because the *square* of the quark form factor appears in eq. (5.20).

where [10]

$$\Gamma_{\rm SDG}(u) = u\int_0^\infty \frac{dy}{y}\ddot{\mathcal{V}}_s(y)\exp(-u\ln y) = \left(\frac{\pi u}{\sin\pi u}\right)^2\frac{1}{(1-u)(1-u/2)}\;. \tag{5.25}$$

The left-hand side of eq. (5.24) is easily obtained to order u^2 from eq. (2.33), in the large-β_0 limit. Our definition of the Borel transform is such that if $f(a_s) = f_1 a_s + f_2 a_s^2 + f_3 a_s^3 + \ldots$, then $B[f](u) = f_1 + \frac{f_2}{\beta_0}u + \frac{1}{2}\frac{f_3}{\beta_0^2}u^2 + \ldots$, with $f(a_s) = \frac{1}{\beta_0}\int_0^\infty du\exp(-u/(\beta_0 a_s))B[f](u)$.

5.2 DY case

We start from the dispersive representation

$$\left.\frac{d\ln\sigma_{\rm DY}(Q^2,N,\mu^2)}{d\ln Q^2}\right|_{\text{large-}\beta_0} = \int_0^\infty \frac{d\lambda^2}{\lambda^2}\ddot{\mathcal{F}}_{\rm DY}\left(\frac{\lambda^2}{Q^2},N\right)A^V_{\rm Mink}(\lambda^2)\,, \tag{5.26}$$

and take the large-N limit, using the following two scaling properties:

(i) For $N\to\infty$ with $\epsilon_s^2 \equiv N^2\epsilon$ fixed, we have [3, 10, 11] (see appendix B)

$$\ddot{\mathcal{F}}_{\rm DY}(\epsilon,N) \sim \ddot{\mathcal{G}}_{\rm DY}(\epsilon_s^2)\;. \tag{5.27}$$

(ii) For $N\to\infty$ with ϵ fixed we have (see eq. (B.12))

$$\ddot{\mathcal{F}}_{\rm DY}(\epsilon,N) \sim \ddot{\mathcal{V}}_t(\epsilon)\,, \tag{5.28}$$

where [24]

$$\mathcal{V}_t(\epsilon) = Re\;\mathcal{V}_s(-\epsilon) = -\int_0^1 dz\frac{(1-z)^2}{z+\epsilon}\ln\frac{z}{\epsilon}\;. \tag{5.29}$$

Proceeding as in section 5.1, we then obtain immediately at large-N the analogues of eq. (5.12) and (5.18):

$$\left.\frac{d\ln\sigma_{\rm DY}(Q^2,N,\mu^2)}{d\ln Q^2}\right|_{\text{large-}\beta_0} \sim \int_0^\infty\frac{d\lambda^2}{\lambda^2}\ddot{\mathcal{G}}_{\rm DY}(N^2\epsilon)A^V_{\rm Mink}(\lambda^2) + \int_0^\infty\frac{d\lambda^2}{\lambda^2}\left[\ddot{\mathcal{V}}_t(\epsilon)+1\right]A^V_{\rm Mink}(\lambda^2)\,, \tag{5.30}$$

and

$$\left.\frac{d\ln\sigma_{\rm DY}(Q^2,N,\mu^2)}{d\ln Q^2}\right|_{\text{large-}\beta_0} \sim \int_0^\infty\frac{d\lambda^2}{\lambda^2}\left[\ddot{\mathcal{G}}_{\rm DY}(N^2\epsilon)+1\right]A^V_{\rm Mink}(\lambda^2) + \int_0^\infty\frac{d\lambda^2}{\lambda^2}\ddot{\mathcal{V}}_t(\epsilon)A^V_{\rm Mink}(\lambda^2)\;. \tag{5.31}$$

The result for the time-like form factor is immediately obtained by analytic continuation of eq. (5.20) to the time-like region. Performing this continuation and taking the real part of the result one gets the formal IR divergent dispersive representation

$$\left.\frac{d\ln|\mathcal{F}(-Q^2)|^2}{d\ln Q^2}\right|_{\text{large-}\beta_0} = \int_0^\infty\frac{d\lambda^2}{\lambda^2}\ddot{\mathcal{V}}_t(\epsilon)A^V_{\rm Mink}(\lambda^2)\,, \tag{5.32}$$

whereas the analogue of eq. (5.17) is clearly

$$\left.\frac{d^2\ln|\mathcal{F}(Q^2)|^2}{(d\ln Q^2)^2}\right|_{\text{large-}\beta_0} = -\int_0^\infty\frac{d\lambda^2}{\lambda^2}\frac{d^3\mathcal{V}_t(\epsilon)}{(d\ln\epsilon)^3}A^V_{\rm Mink}(\lambda^2)\;. \tag{5.33}$$

6. Conclusion

In this paper we investigated the structure of N-independent contributions in threshold resummation for DIS and the DY process. Our main result is contained in the conjectured relations eq. (2.26) and (3.15), which have been checked to order α_s^4, using the relations eq. (2.39) and (3.20). These relations essentially state that, once one corrects for the mismatch due to the presence of a virtual contribution (required to regulate IR divergencies) in the Sudakov integrals (which ideally should contain only real gluon emission contributions responsible for the logarithmic terms at large-N), the remaining constant terms not included in the integrals are given by (logarithmic derivatives of) the quark form factor, a purely virtual contribution. To obtain these relations, two derivatives are necessary (eq. (2.24) and (3.14)): the first one gets rid of infrared and collinear divergences, leading to the IR safe "physical anomalous dimension" observable; the second one allows to bypass the real-virtual cancellation of IR singularities, the purely real contributions (the Sudakov integrals in eq. (2.24) and (3.14)) and the remainder, purely virtual, form factor related constant terms being *separately* finite.

The close connection between N-independent ("non-logarithmic") terms and form factor type contributions have been noted for a long time [1, 17, 18, 21, 26]. We presented a particularly simple version of this connection, valid in four dimensions, where the DIS and DY channels are treated in a symmetrical way. As a by-product, we obtained eq. (2.90) and (3.48), which allow to compute the "non-conformal" parts B and D of the standard "jet" and "soft" Sudakov effective charges $\mathcal{J}$ and $\mathcal{S}$ in terms of the virtual contribution B_δ to the diagonal splitting function, and the quark form factor. While the second of these relations has a content equivalent to similar ones previously given in [17, 18, 21] for the DY process (which allow in particular to compute D_3, yielding a result which agrees with the one obtained in [17, 18, 27, 28]), its counterpart eq. (2.90) for the DIS case is new, and allows to compute B_3 with a method alternative to the one used in [5]. Moreover, subtracting eq. (2.90) from (3.48), we obtained the general structure (eq. (3.53)) laying behind the first relation in eq. (4.19) of [5], which we extended to one more order (eq. (3.55)).

We also performed an all-order check of our conjecture at large-n_f, taking the large-N limit of the dispersive representation of the "physical anomalous dimensions" which control the scaling violation for DIS and DY. As a by-product, we obtained a dispersive representation of the quark form factor.

A further consequence of eq. (2.26) and (3.15) was pointed out in [3]: if the theory is conformal in the IR limit, these relations imply *universality* of the IR fixed points of the DIS and DY Sudakov effective couplings $\mathcal{J}(k^2)$ and $\mathcal{S}(k^2)$; in particular, the Banks-Zaks fixed points are the same (and independent of the resummation procedure, i.e. the same for $\mathcal{J}_{\rm new}(k^2)$ and $\mathcal{S}_{\rm new}(k^2)$).

Although we investigated only DIS and DY, we expect a similar approach to be applicable to other inclusive processes with a hard electromagnetic vertex, such as the ones considered in [29], as well as [18] to the case, relevant to Higgs production via gluon fusion, where the hard vertex is a gluon form factor.

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A. The large-N constant terms in the Sudakov integral

To compute the constant terms c_p in eq. (2.53), *i.e* the terms proportional to N^0 (or, rather, N^{-0}) in the $N \to \infty$ asymptotic expansions of (2.53), it is very convenient to introduce the Mellin-Barnes representation

$$\exp\left(-\frac{Nk^2}{Q^2}\right) - 1 = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{Nk^2}{Q^2}\right)^{-s} \Gamma(s)\,, \tag{A.1}$$

with $c \equiv \Re(s) \in]-1, 0[$. The latter interval defines the s-complex plane fundamental strip of the Mellin-Barnes representation, crucial object to determine the asymptotic expansion. Indeed, the Mellin transform singularities lying to the right of the fundamental strip encode the $N \to \infty$ asymptotic expansion of the Mellin-Barnes representation, while the singularities to the left encode the $N \to 0$ asymptotic expansion. It is therefore important to know precisely the fundamental strip in order to take into account only the relevant singularities (for more details, in particular for the determination of the fundamental strip, we refer the reader to [30, 31]).

Using (A.1) in eq. (2.53), we then have

$$\begin{aligned} I_0(N) &\equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, N^{-s}\tilde{I}_0(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{N}{Q^2}\right)^{-s}\Gamma(s)\int_0^{Q^2} dk^2 \frac{1}{(k^2)^{1+s}} \\ &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, N^{-s}\Gamma(s)\frac{1}{s}\,, \end{aligned} \tag{A.2}$$

$$\begin{aligned} I_1(N) &\equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, N^{-s}\tilde{I}_1(s) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{N}{Q^2}\right)^{-s}\Gamma(s)\int_0^{Q^2} dk^2 \frac{1}{(k^2)^{1+s}}\ln\left(\frac{k^2}{Q^2}\right) \\ &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, N^{-s}\Gamma(s)\frac{1}{s^2} \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} I_2(N) &\equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, N^{-s}\tilde{I}_2(s) \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{N}{Q^2}\right)^{-s}\Gamma(s)\int_0^{Q^2} dk^2 \frac{1}{(k^2)^{1+s}}\ln^2\left(\frac{k^2}{Q^2}\right) \\ &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, N^{-s}\Gamma(s)\frac{2}{s^3}\,, \end{aligned} \tag{A.4}$$

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where the last step of integration for each integral is true only if $\Re(s) < 0$. Since $\Re(s) \in]-1,0[$, this is actually the case. For arbitrary p we have

$$I_p(N) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, N^{-s}\tilde{I}_p(s) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, N^{-s}\Gamma(s)\frac{p\,!}{s^{p+1}}\,, \tag{A.5}$$

with the same condition $\Re(s) < 0$.

Now, we are interested in the $N \to \infty$ asymptotic expansion of these integrals and, to be precise, in the constant term of the asymptotic expansion, therefore we only have to consider the corresponding singular element of the $\tilde{I}_p(s)$ (to the right of their fundamental strip). This is indeed the statement of the *converse mapping theorem* [30], which gives nothing but a simple dictionary between singular elements (or singularities) of the Mellin transforms (here the $\tilde{I}_p(s)$) and terms of the asymptotic expansions of their inverse (here the $I_p(N)$). In our case, the singular elements of interest $\tilde{I}_p|^{sing.}$ ($p \in \{0,1,2\}$) are the truncated Laurent series of each $\tilde{I}_p(s)$ around $s=0$, because the *converse mapping theorem* relates the residue of a pole located at $s=l$ to the coefficient of the term of power $-l$ in the asymptotic expansion (which is for us N^{-0}).

The singular elements at $s=0$ are

$$\begin{aligned} \tilde{I}_0|^{sing.}_{s\to 0} &= -\frac{1}{s^2} + \frac{\gamma_E}{s}\,, \\ \tilde{I}_1|^{sing.}_{s\to 0} &= -\frac{1}{s^3} + \frac{\gamma_E}{s^2} - \frac{6\gamma_E^2+\pi^2}{12s} \end{aligned} \tag{A.6}$$

and

$$\tilde{I}_2|^{sing.}_{s\to 0} = -\frac{2}{s^4} + \frac{2\gamma_E}{s^3} - \frac{6\gamma_E^2+\pi^2}{6s^2} + \frac{2\gamma_E^3+\gamma_E\pi^2+4\zeta_3}{6s}\,. \tag{A.7}$$

Since the Mellin transforms $\tilde{I}_p(s)$ fulfil the necessary condition of decrease along vertical lines [30], we can apply the *converse mapping theorem*[11] and we then find the results in eqs. (2.55), (2.56) and (2.57). Notice that with this method any term of the asymptotic expansions can be straightforwardly obtained by computing the corresponding singular element (see appendix C for an example of a complete asymptotic expansion computation).

B. Scaling behavior of the characteristic functions in the large-N limit

B.1 Deep Inelastic Scattering

We start from the expression eq. (5.9) for the Mellin-space characteristic function $\mathcal{F}_{\rm DIS}(\epsilon,N)$, and derive its $N\to\infty$ limit, with $\epsilon_j \equiv N\epsilon = \frac{N\lambda^2}{Q^2}$ fixed. Let us first consider the real contribution

$$\mathcal{F}^{(r)}_{\rm DIS}(\epsilon,N) = \int_0^{\frac{1}{1+\epsilon}} dx\; x^{N-1}\tilde{\mathcal{F}}^{(r)}_{\rm DIS}(\epsilon,x)\,. \tag{B.1}$$

[11] The *converse mapping theorem* says that a pole of multiplicity m located at $s=l$ gives a term proportional to $N^{-l}\ln^{m-1}(N)$ so that we only need to consider the $m=1$ terms in our singular elements.

Using the change of variable $t = N(1-x)$, we get

$$\mathcal{F}^{(r)}_{\rm DIS}(\epsilon, N) = \int_{\frac{\epsilon_j}{1+\frac{\epsilon_j}{N}}}^{N} dt \left(1 - \frac{t}{N}\right)^{N-1} \frac{1}{N}\tilde{\mathcal{F}}^{(r)}_{\rm DIS}\left(\frac{\epsilon_j}{N}, 1 - \frac{t}{N}\right) . \tag{B.2}$$

Now for $N \to \infty$, $\left(1 - \frac{t}{N}\right)^{N-1} \sim \exp(-t)$, whereas, using the expression of $\tilde{\mathcal{F}}^{(r)}_{\rm DIS}(\epsilon, x)$ given in [24], one finds[12]

$$\frac{1}{N}\tilde{\mathcal{F}}^{(r)}_{\rm DIS}\left(\frac{\epsilon_j}{N}, 1 - \frac{t}{N}\right) \sim \frac{1}{t}\left(\ln\frac{t}{\epsilon_j} - \frac{3}{4} + \frac{1}{2}\frac{\epsilon_j}{t} + \frac{1}{4}\frac{\epsilon_j^2}{t^2}\right) , \tag{B.3}$$

(where we accounted for the different normalization by a factor 1/2). Thus taking the limit $N \to \infty$ with ϵ_j fixed we get

$$\mathcal{F}^{(r)}_{\rm DIS}(\epsilon, N) \sim \int_{\epsilon_j}^{+\infty} \frac{dt}{t} \, \exp(-t) \left[\ln\frac{t}{\epsilon_j} - \frac{3}{4} + \frac{1}{2}\frac{\epsilon_j}{t} + \frac{1}{4}\frac{\epsilon_j^2}{t^2}\right] , \tag{B.4}$$

where we set $N = \infty$ in the limits of integration. We are interested in the behavior of the second derivative with respect to $\ln\epsilon_j$. It is straightforward to get from eq. (B.4)

$$\dot{\mathcal{F}}^{(r)}_{\rm DIS}(\epsilon, N) \sim \int_{\epsilon_j}^{+\infty} \frac{dt}{t} \, \exp(-t) \left[1 - \frac{1}{2}\frac{\epsilon_j}{t} - \frac{1}{2}\frac{\epsilon_j^2}{t^2}\right] \tag{B.5}$$

and

$$\begin{aligned}\ddot{\mathcal{F}}^{(r)}_{\rm DIS}(\epsilon, N) &\sim \int_{\epsilon_j}^{+\infty} \frac{dt}{t} \, \exp(-t) \left[\frac{1}{2}\frac{\epsilon_j}{t} + \frac{\epsilon_j^2}{t^2}\right] \\ &= \frac{\epsilon_j}{2}\Gamma(-1, \epsilon_j) + \epsilon_j^2\Gamma(-2, \epsilon_j) ,\end{aligned} \tag{B.6}$$

where we recall that $\dot{\mathcal{F}} \equiv -\frac{d\mathcal{F}}{d\ln\lambda^2} = +\frac{d\mathcal{F}}{d\ln Q^2}$. On the other hand, for $\epsilon = \epsilon_j/N \to 0$ the virtual contribution $\mathcal{V}_s(\epsilon)$ behaves as

$$\mathcal{V}_s(\epsilon) \sim -\frac{1}{2}\ln^2\epsilon - \frac{3}{2}\ln\epsilon - \frac{\pi^2}{3} - \frac{7}{4} , \tag{B.7}$$

and thus diverges for $N \to \infty$ with ϵ_j fixed, but this divergence is removed after taking two derivatives, since (see also eq. (5.13)) $\ddot{\mathcal{V}}_s(0) = -1$. We can express the incomplete Gamma functions $\Gamma(-1, \epsilon_j)$ and $\Gamma(-2, \epsilon_j)$ on the right-hand side of eq. (B.6) in terms of $\Gamma(0, \epsilon_j)$ using integration by parts

$$\Gamma(-1, x) = \frac{\exp(-x)}{x} - \Gamma(0, x) \tag{B.8}$$

[12]We note that after multiplication by t, and reverting to the original variables x and ϵ, the left-hand side of eq. (B.3) coincides with $(1-x)\tilde{\mathcal{F}}^{(r)}_{\rm DIS}(\epsilon, x)$. We thus find for $x \to 1$ with $\frac{\epsilon}{1-x}$ fixed the scaling law in momentum space $(1-x)\tilde{\mathcal{F}}^{(r)}_{\rm DIS}(\epsilon, x) \sim \ln\frac{1-x}{\epsilon} - \frac{3}{4} + \frac{1}{2}\frac{\epsilon}{1-x} + \frac{1}{4}(\frac{\epsilon}{1-x})^2$, where the right-hand side depends only on the *single* variable $\frac{\epsilon}{1-x} = \frac{\lambda^2}{Q^2(1-x)}$, and coincides (taking into account the different normalization) with $(1-x)\mathcal{F}(x, \epsilon)\,|_{\rm log}$ in the notation of [32] (see eq. (28) there).

and

$$\Gamma(-2,x) = \frac{1}{2}\frac{\exp(-x)}{x^2} - \frac{1}{2}\frac{\exp(-x)}{x} + \frac{1}{2}\Gamma(0,x)\,, \tag{B.9}$$

to obtain

$$\ddot{\mathcal{F}}^{(r)}_{\rm DIS}(\epsilon,N) \sim \exp(-\epsilon_j) - \frac{1}{2}\epsilon_j \exp(-\epsilon_j) - \frac{1}{2}\epsilon_j\Gamma(0,\epsilon_j) + \frac{1}{2}\epsilon_j^2\Gamma(0,\epsilon_j)\ . \tag{B.10}$$

Hence, adding the virtual contribution $\ddot{\mathcal{V}}_s(0) = -1$, we get [3, 10, 11]

$$\ddot{\mathcal{F}}_{\rm DIS}(\epsilon,N) \sim -1+\exp(-\epsilon_j) - \frac{1}{2}\epsilon_j \exp(-\epsilon_j) - \frac{1}{2}\epsilon_j\Gamma(0,\epsilon_j) + \frac{1}{2}\epsilon_j^2\Gamma(0,\epsilon_j) \equiv \ddot{\mathcal{G}}_{\rm DIS}(\epsilon_j)\ . \tag{B.11}$$

B.2 Drell-Yan

Exactly the same reasoning as for DIS can be applied to DY. As before, we use results of [24] at finite N to compute the corresponding large-N limit. We begin with the analogue of eq. (5.9)

$$\mathcal{F}_{\rm DY}(\epsilon,N) = \int_0^{\tau_{\rm max}} dx\; x^{N-1}\tilde{\mathcal{F}}^{(r)}_{\rm DY}(\epsilon,x) + \mathcal{V}_t(\epsilon)\,, \tag{B.12}$$

where $\tau_{\rm max} = \frac{1}{(1+\sqrt{\epsilon})^2}$, and derive the $N\to\infty$ limit of $\mathcal{F}_{\rm DY}(\epsilon,N)$, with $\epsilon_s \equiv N\sqrt{\epsilon} = \frac{N\lambda}{Q}$ fixed. Using again the change of variable $t = N(1-x)$, we get for the real contribution

$$\mathcal{F}^{(r)}_{\rm DY}(\epsilon,N) = \int_{\frac{2\epsilon_s+\frac{\epsilon_s^2}{N}}{\left(1+\frac{\epsilon_s}{N}\right)^2}}^{N} dt\left(1-\frac{t}{N}\right)^{N-1}\frac{1}{N}\tilde{\mathcal{F}}^{(r)}_{\rm DY}\left(\frac{\epsilon_s^2}{N^2}, 1-\frac{t}{N}\right)\ . \tag{B.13}$$

Starting from the expression of $\tilde{\mathcal{F}}^{(r)}_{\rm DY}(\epsilon,x)$ given in [24], it is easy to show that for $N\to\infty$ we have[13] (taking into account the different normalization by a factor of 1/2)

$$\frac{1}{N}\tilde{\mathcal{F}}^{(r)}_{\rm DY}\left(\frac{\epsilon_s^2}{N^2}, 1-\frac{t}{N}\right) \sim \frac{4}{t}\tanh^{-1}\sqrt{1-\frac{4\epsilon_s^2}{t^2}}\ . \tag{B.14}$$

Thus letting $N\to\infty$ with ϵ_s fixed we get

$$\mathcal{F}^{(r)}_{\rm DY}(\epsilon,N) \sim 4\int_{2\epsilon_s}^{+\infty}\frac{dt}{t}\,\exp(-t)\,\tanh^{-1}\sqrt{1-\frac{4\epsilon_s^2}{t^2}}\,, \tag{B.15}$$

where we set $N=\infty$ in the limits of integration. Taking (minus) the first derivative with respect to $\ln\epsilon_s^2$ of eq. (B.15) we obtain

$$\dot{\mathcal{F}}^{(r)}_{\rm DY}(\epsilon,N) \sim 2\int_{2\epsilon_s}^{+\infty}\frac{dt}{t}\,\exp(-t)\,\frac{1}{\sqrt{1-\frac{4\epsilon_s^2}{t^2}}} = 2K_0\left(2\epsilon_s\right)\,, \tag{B.16}$$

[13] Again we note that after multiplication by t, and reverting to the original variables x and ϵ, the left-hand side of eq. (B.14) coincides with $(1-x)\tilde{\mathcal{F}}^{(r)}_{\rm DY}(\epsilon,x)$. We thus find for $x\to 1$ with $\frac{\epsilon}{(1-x)^2}$ fixed the scaling law in momentum space $(1-x)\tilde{\mathcal{F}}^{(r)}_{\rm DY}(\epsilon,x) \sim 4\tanh^{-1}\sqrt{1-4\frac{\epsilon}{(1-x)^2}}$, where the right-hand side depends only on the *single* variable $\frac{\epsilon}{(1-x)^2} = \frac{\lambda^2}{Q^2(1-x)^2}$.

where K_0 is the modified Bessel function of the second kind.[14] On the other hand, for $\epsilon = \epsilon_s^2/N^2 \to 0$ the virtual contribution $\mathcal{V}_t(\epsilon)$ behaves as

$$\mathcal{V}_t(\epsilon) \sim -\frac{1}{2}\ln^2\epsilon - \frac{3}{2}\ln\epsilon + \frac{\pi^2}{6} - \frac{7}{4}, \tag{B.17}$$

and thus diverges for $N \to \infty$ with ϵ_s fixed, but this divergence is again removed after taking two derivatives, and we get $\ddot{\mathcal{V}}_t(0) = -1$. Thus we obtain

$$\ddot{\mathcal{F}}_{\rm DY}(\epsilon, N) \sim -\frac{d}{d\ln\epsilon_s^2}\left[2K_0\left(2\epsilon_s\right)\right] - 1 = -\frac{d}{d\ln x}\left[K_0\left(x = 2\epsilon_s\right)\right] - 1 \equiv \ddot{\mathcal{G}}_{\rm DY}(\epsilon_s^2), \tag{B.18}$$

which agrees with the result quoted in [3, 10, 11].

C. Massless one-loop quark form factor with a finite gluon mass

Let us detail the calculation of the massless one-loop renormalized quark form factor with a finite gluon mass λ. We present here this calculation in dimensional regularisation $D = 4-\epsilon$ and in Feynman gauge (the "Landau gauge" $k_\mu k_\nu$ term gives no contribution).

The amplitude we are interested in is

$$A = (-ie)(-i)g^2 t^a t^a \mu^\epsilon \bar{u}(p+q)\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}}\,\gamma^\mu \frac{1}{\hat{k}+\hat{p}+\hat{q}+i\eta}\gamma_\nu \frac{1}{\hat{k}+\hat{p}+i\eta}\gamma_\mu \frac{1}{k^2-\lambda^2+i\eta}\,u(p) \tag{C.1}$$

with $t^a t^a = C_F = \frac{4}{3}$. In the (on-shell) massless quark limit only one form factor enters into the game:

$$A = (-ie)\mathcal{F}_1(Q^2,\epsilon)\bar{u}(p+q)\gamma_\nu u(p), \tag{C.2}$$

where $Q^2 \equiv -q^2$.

Feynman parametrisation, gamma-algebra and evaluation of the momentum integral lead to (we do not write $i\eta$ anymore)

$$\begin{aligned}\mathcal{F}_1(Q^2,\epsilon) \;=\; &(-i)\alpha_s 4\pi C_F\left\{\mu^\epsilon 2\frac{i}{(2\sqrt{\pi})^{4-\epsilon}}\frac{\Gamma(3-\frac{\epsilon}{2})\Gamma(\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})\Gamma(3)}\frac{(\epsilon-2)^2}{4-\epsilon}\right.\\ &\times\int_0^1 dx\int_0^1 dyy[Q^2xy^2(1-x)+\lambda^2(1-y)]^{\frac{-\epsilon}{2}} + \mu^\epsilon 2\frac{-i}{(2\sqrt{\pi})^{4-\epsilon}}\frac{\Gamma(1+\frac{\epsilon}{2})}{\Gamma(3)}\\ &\left.\times\int_0^1 dx\int_0^1 dyy[Q^2xy^2(1-x)+\lambda^2(1-y)]^{-1-\frac{\epsilon}{2}}Q^2[2(1-y)+(x-1)xy^2(\epsilon-2)]\right\}.\end{aligned} \tag{C.3}$$

The exact values of these integrals are not straightforwardly computed but one can get their asymptotic expansions in the $\lambda \to 0$ limit. Interestingly enough, we shall see that exact results can be obtained from the asymptotic expansions. The calculation will be done following the strategy of [30, 31] which has already been used and detailed in appendix A.

[14] A connection with the work in [33] was pointed out in [3].

Let us consider the first parametric integral in the bracket of (C.3). It reads

$$I_1 \equiv \mu^\epsilon 2\frac{i}{(2\sqrt{\pi})^{4-\epsilon}}\frac{\Gamma(3-\frac{\epsilon}{2})\Gamma(\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})\Gamma(3)}\frac{(\epsilon-2)^2}{4-\epsilon}\int_0^1 dx\int_0^1 dyy[Q^2xy^2(1-x)+\lambda^2(1-y)]^{\frac{-\epsilon}{2}}$$
$$= \mu^\epsilon 2\frac{i}{(2\sqrt{\pi})^{4-\epsilon}}\frac{\Gamma(3-\frac{\epsilon}{2})\Gamma(\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})\Gamma(3)}\frac{(\epsilon-2)^2}{4-\epsilon}\int_0^1 dx\int_0^1 dy\ y\frac{[Q^2xy^2(1-x)]^{\frac{-\epsilon}{2}}}{\left[1+\frac{\lambda^2(1-y)}{Q^2xy^2(1-x)}\right]^{\frac{\epsilon}{2}}}\ . \tag{C.4}$$

Using the Mellin-Barnes representation

$$\frac{1}{\left[1+\frac{\lambda^2(1-y)}{Q^2xy^2(1-x)}\right]^{\frac{\epsilon}{2}}} = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} ds\left(\frac{\lambda^2(1-y)}{Q^2xy^2(1-x)}\right)^{-s}\frac{\Gamma(s)\Gamma\left(\frac{\epsilon}{2}-s\right)}{\Gamma\left(\frac{\epsilon}{2}\right)}\,, \tag{C.5}$$

with $c \equiv \Re(s) \in]0,\frac{\epsilon}{2}[$ (the s-complex plane fundamental strip of the Mellin-Barnes representation), we then get

$$I_1 = \mu^\epsilon 2\frac{i}{(2\sqrt{\pi})^{4-\epsilon}}\frac{\Gamma(3-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})\Gamma(3)}\frac{(\epsilon-2)^2}{4-\epsilon}(Q^2)^{-\frac{\epsilon}{2}}$$
$$\times\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} ds\left(\frac{\lambda^2}{Q^2}\right)^{-s}\frac{\pi}{\sin(\pi s)}\frac{\Gamma\left(\frac{\epsilon}{2}-s\right)\Gamma^2\left(1-\frac{\epsilon}{2}+s\right)}{\Gamma\left(3-\epsilon+s\right)}\,, \tag{C.6}$$

where we performed the parametric integrals, which did not modify the fundamental strip.

Similarly, using

$$\frac{1}{\left[1+\frac{\lambda^2(1-y)}{Q^2xy^2(1-x)}\right]^{1+\frac{\epsilon}{2}}} = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} ds\left(\frac{\lambda^2(1-y)}{Q^2xy^2(1-x)}\right)^{-s}\frac{\Gamma(s)\Gamma\left(1+\frac{\epsilon}{2}-s\right)}{\Gamma\left(1+\frac{\epsilon}{2}\right)}\ , \tag{C.7}$$

where the fundamental strip is given in this case by $c \in]0, 1+\frac{\epsilon}{2}[$, we have for the second integral in the bracket of (C.3)

$$I_2 \equiv \mu^\epsilon 2\frac{-i}{(2\sqrt{\pi})^{4-\epsilon}}\frac{\Gamma(1+\frac{\epsilon}{2})}{\Gamma(3)}$$
$$\times\int_0^1 dx\int_0^1 dy\ y[Q^2xy^2(1-x)+\lambda^2(1-y)]^{-1-\frac{\epsilon}{2}}Q^2[2(1-y)+(x-1)xy^2(\epsilon-2)]$$
$$= \mu^\epsilon 2\frac{-i}{(2\sqrt{\pi})^{4-\epsilon}}\frac{\Gamma(1+\frac{\epsilon}{2})}{\Gamma(3)}\left\{2Q^2\int_0^1 dx\int_0^1 dy\ y(1-y)[Q^2xy^2(1-x)+\lambda^2(1-y)]^{-1-\frac{\epsilon}{2}}\right.$$
$$\left.+(\epsilon-2)Q^2\int_0^1 dx\int_0^1 dy\ (x-1)xy^3[Q^2xy^2(1-x)+\lambda^2(1-y)]^{-1-\frac{\epsilon}{2}}\right\}$$
$$= \mu^\epsilon 2\frac{-i}{(2\sqrt{\pi})^{4-\epsilon}}\frac{\Gamma(1+\frac{\epsilon}{2})}{\Gamma(3)}(Q^2)^{-\frac{\epsilon}{2}}$$
$$\times\left(2\frac{1}{2i\pi}\int_{d-i\infty}^{d+i\infty} ds\left(\frac{\lambda^2}{Q^2}\right)^{-s}\frac{\Gamma(s)\Gamma\left(1+\frac{\epsilon}{2}-s\right)}{\Gamma\left(1+\frac{\epsilon}{2}\right)}\frac{\Gamma^2\left(s-\frac{\epsilon}{2}\right)\Gamma(2-s)}{\Gamma\left(s+2-\epsilon\right)}\right.$$
$$\left.-(\epsilon-2)\frac{1}{2i\pi}\int_{f-i\infty}^{f+i\infty} ds\left(\frac{\lambda^2}{Q^2}\right)^{-s}\frac{\Gamma(s)\Gamma\left(1+\frac{\epsilon}{2}-s\right)}{\Gamma\left(1+\frac{\epsilon}{2}\right)}\frac{\Gamma^2\left(1-\frac{\epsilon}{2}+s\right)\Gamma(1-s)}{\Gamma\left(s+3-\epsilon\right)}\right)\,. \tag{C.8}$$

Notice that for the two Mellin-Barnes integrals in the last equation, the fundamental strips have been modified by the parametric integrations, since $d \in]\frac{\epsilon}{2}, 1+\frac{\epsilon}{2}[$ and $f \in]0,1[$.

It is possible to compute the asymptotic expansion of (C.6) and (C.8) in the $\lambda \to 0$ limit keeping an exact dependance in ϵ, but we would get results on which the ϵ Laurent expansion would be hard to obtain. In fact both integrals in (C.8) are convergent in the $\epsilon \to 0$ limit, which can then be performed at the integrand level in these integrals to give

$$I_2|_{\epsilon\to 0} = -\frac{i}{8\pi^2}\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{\lambda^2}{Q^2}\right)^{-s}\left(\frac{\pi}{\sin(\pi s)}\right)^2 \frac{2-s}{s(s+1)(s+2)}\,, \tag{C.9}$$

where the fundamental strip is now $c \in]0,1[$.

A contrario, the $\epsilon \to 0$ limit of (C.6) is not well-defined since we have a pinch singularity in this limit (due to the fundamental strip $]0,\frac{\epsilon}{2}[$). This, of course, reflects the UV divergence of the form factor.

What we therefore do is to keep the exact ϵ-dependance to compute the first term of the $\lambda \to 0$ asymptotic expansion of (C.6). After that, the pinch singularity being discarded, the ϵ-expansion becomes possible in the remainder integral directly at the integrand level. We then perform renormalization, subtracting to the form factor its value at $Q^2 = 0$.

To compute the first term of the $\lambda \to 0$ asymptotic expansion of (C.6), we follow [30, 31] (see also appendix A). One thus needs the first singular element to the left of the fundamental strip of (C.6), which is located at $s = 0$ and reads

$$\left[\frac{\pi}{\sin(\pi s)}\frac{\Gamma\left(\frac{\epsilon}{2}-s\right)\Gamma^2\left(1-\frac{\epsilon}{2}+s\right)}{\Gamma\left(3-\epsilon+s\right)}\right]\Bigg|_{s\to 0}^{sing.} = \frac{1}{s}\frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma^2\left(1-\frac{\epsilon}{2}\right)}{\Gamma\left(3-\epsilon\right)}\,. \tag{C.10}$$

We therefore conclude that

$$\begin{aligned} I_1 &= \mu^\epsilon 2\frac{i}{(2\sqrt{\pi})^{4-\epsilon}}\frac{\Gamma(3-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})\Gamma(3)}\frac{(\epsilon-2)^2}{4-\epsilon}(Q^2)^{-\frac{\epsilon}{2}}\frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma^2\left(1-\frac{\epsilon}{2}\right)}{\Gamma\left(3-\epsilon\right)} \\ &\quad+\mu^\epsilon 2\frac{i}{(2\sqrt{\pi})^{4-\epsilon}}\frac{\Gamma(3-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})\Gamma(3)}\frac{(\epsilon-2)^2}{4-\epsilon}(Q^2)^{-\frac{\epsilon}{2}} \\ &\quad\times\frac{1}{2i\pi}\int_{d-i\infty}^{d+i\infty} ds\left(\frac{\lambda^2}{Q^2}\right)^{-s}\frac{\pi}{\sin(\pi s)}\frac{\Gamma\left(\frac{\epsilon}{2}-s\right)\Gamma^2\left(1-\frac{\epsilon}{2}+s\right)}{\Gamma\left(3-\epsilon+s\right)}\,, \end{aligned} \tag{C.11}$$

where now $d \in]-1+\frac{\epsilon}{2}, 0[$, so that one can safely perform the $\epsilon \to 0$ limit inside the integral. Before doing this, let us come back to eq. (C.3) to compute the contribution of renormalization.

$$\begin{aligned} \mathcal{F}_1(0,\epsilon) &= (-i)\alpha_s 4\pi C_F 2\frac{i}{(2\sqrt{\pi})^{4-\epsilon}}\mu^\epsilon\frac{\Gamma(3-\frac{\epsilon}{2})\Gamma(\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})\Gamma(3)}\frac{(\epsilon-2)^2}{4-\epsilon}\int_0^1 dx\int_0^1 dy\; y[\lambda^2(1-y)]^{\frac{-\epsilon}{2}} \\ &= (-i)\alpha_s 4\pi C_F 2\frac{i}{(2\sqrt{\pi})^{4-\epsilon}}\mu^\epsilon\frac{\Gamma(3-\frac{\epsilon}{2})\Gamma(\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})\Gamma(3)}\frac{4(2-\epsilon)}{(4-\epsilon)^2}\lambda^{-\epsilon}\,. \end{aligned} \tag{C.12}$$

One therefore finds

$$
\begin{aligned}
\mathcal{F}_{1,R}(Q^2) \equiv\; & \left(\mathcal{F}_1(Q^2,\epsilon)-\mathcal{F}_1(0,\epsilon)\right)\Big|_{\epsilon=0} \\
=\; & (-i)\alpha_s 4\pi C_F \left(I_1+I_2-2\frac{i}{(2\sqrt{\pi})^{4-\epsilon}}\mu^{\epsilon}\frac{\Gamma(3-\frac{\epsilon}{2})\Gamma(\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})\Gamma(3)}\frac{4(2-\epsilon)}{(4-\epsilon)^2}\lambda^{-\epsilon}\right)\Bigg|_{\epsilon=0} \\
=\; & (-i)\alpha_s 4\pi C_F \left\{\frac{i}{32\pi^2}\left[3+2\ln\left(\frac{\lambda^2}{Q^2}\right)\right]\right. \\
& -\frac{i}{8\pi^2}\frac{1}{2i\pi}\int_{d-i\infty}^{d+i\infty} ds \left(\frac{\lambda^2}{Q^2}\right)^{-s}\left(\frac{\pi}{\sin(\pi s)}\right)^2\frac{1}{(2+s)(1+s)} \\
& \left.-\frac{i}{8\pi^2}\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{\lambda^2}{Q^2}\right)^{-s}\left(\frac{\pi}{\sin(\pi s)}\right)^2\frac{2-s}{(2+s)(1+s)s}\right\},
\end{aligned}
\tag{C.13}
$$

with $d \in]-1,0[$ and $c \in]0,1[$. To get the complete $\lambda \to 0$ asymptotic expansions of the two integrals in the right-hand side of (C.14), we then have to compute the singular expansion[15] of the Mellin transforms

$$
\mathcal{M}_1(s) \equiv \left(\frac{\pi}{\sin(\pi s)}\right)^2 \frac{1}{(2+s)(1+s)} \tag{C.14}
$$

and

$$
\mathcal{M}_2(s) \equiv \left(\frac{\pi}{\sin(\pi s)}\right)^2 \frac{2-s}{s(s+1)(s+2)} \tag{C.15}
$$

to the left of their corresponding fundamental strips ($i.e$ $]-1,0[$ for $\mathcal{M}_1$ and $]0,1[$ for $\mathcal{M}_2$).

One finds

$$
\begin{aligned}
\mathcal{M}_1(s) \asymp\; & \frac{1}{(s+1)^3}-\frac{1}{(s+1)^2}+\frac{1+\frac{\pi^2}{3}}{s+1}-\frac{1}{(s+2)^3}-\frac{1}{(s+2)^2}-\frac{1+\frac{\pi^2}{3}}{s+2} \\
& +\sum_{n=3}^{\infty}\frac{1}{(2-n)(1-n)}\frac{1}{(s+n)^2}-\sum_{n=3}^{\infty}\frac{3-2n}{(2-n)^2(1-n)^2}\frac{1}{s+n}
\end{aligned}
\tag{C.16}
$$

and

$$
\begin{aligned}
\mathcal{M}_2(s) \asymp\; & \frac{1}{s^3}-\frac{2}{s^2}+\frac{\frac{5}{2}+\frac{\pi^2}{3}}{s}-\frac{3}{(s+1)^3}+\frac{1}{(s+1)^2}-\frac{3+\pi^2}{s+1}+\frac{2}{(s+2)^3}+\frac{5}{2}\frac{1}{(s+2)^2} \\
& +\frac{\frac{11}{4}+\frac{2\pi^2}{3}}{s+2}-\sum_{n=3}^{\infty}\frac{2+n}{(2-n)(1-n)n}\frac{1}{(s+n)^2}+\sum_{n=3}^{\infty}\frac{-4+12n-3n^2-2n^3}{(2-n)^2(1-n)^2n^2}\frac{1}{s+n}.
\end{aligned}
\tag{C.17}
$$

Now, since our Mellin transforms (C.14) and (C.15) fulfil the necessary condition of decrease along vertical lines [30], we can apply the *converse mapping theorem* [30, 31], which gives

[15]The singular expansion is simply the formal sum of all singular elements, and it is denoted by the symbol $\asymp$ as in [30, 31].

the complete asymptotic expansion for $\lambda^2/Q^2 \to 0$

$$\begin{aligned}\mathcal{F}_{1,R}(Q^2) \sim (-i)\alpha_s 4\pi C_F \Bigg\{ &\frac{i}{32\pi^2}\left[3+2\ln\left(\frac{\lambda^2}{Q^2}\right)\right] - \frac{i}{8\pi^2}\left[\frac{1}{2}\frac{\lambda^2}{Q^2}\ln^2\left(\frac{\lambda^2}{Q^2}\right) + \frac{\lambda^2}{Q^2}\ln\left(\frac{\lambda^2}{Q^2}\right)\right.\\
&+\left(1+\frac{\pi^2}{3}\right)\frac{\lambda^2}{Q^2} - \frac{1}{2}\left(\frac{\lambda^2}{Q^2}\right)^2\ln^2\left(\frac{\lambda^2}{Q^2}\right) + \left(\frac{\lambda^2}{Q^2}\right)^2\ln\left(\frac{\lambda^2}{Q^2}\right) - \left(1+\frac{\pi}{3}\right)\left(\frac{\lambda^2}{Q^2}\right)^2\\
&\left.-\ln\left(\frac{\lambda^2}{Q^2}\right)\sum_{n=3}^{\infty}\frac{1}{(2-n)(1-n)}\left(\frac{\lambda^2}{Q^2}\right)^n + \sum_{n=3}^{\infty}\frac{2n-3}{(2-n)^2(1-n)^2}\left(\frac{\lambda^2}{Q^2}\right)^n\right]\\
&-\frac{i}{8\pi^2}\left[\frac{1}{2}\ln^2\left(\frac{\lambda^2}{Q^2}\right) + 2\ln\left(\frac{\lambda^2}{Q^2}\right) + \frac{5}{2} + \frac{\pi^2}{3} - \frac{3}{2}\frac{\lambda^2}{Q^2}\ln^2\left(\frac{\lambda^2}{Q^2}\right) - \frac{\lambda^2}{Q^2}\ln\left(\frac{\lambda^2}{Q^2}\right)\right.\\
&-(3+\pi^2)\frac{\lambda^2}{Q^2} + \left(\frac{\lambda^2}{Q^2}\right)^2\ln^2\left(\frac{\lambda^2}{Q^2}\right) - \frac{5}{2}\left(\frac{\lambda^2}{Q^2}\right)^2\ln\left(\frac{\lambda^2}{Q^2}\right) + \left(\frac{11}{4}+\frac{2\pi^2}{3}\right)\left(\frac{\lambda^2}{Q^2}\right)^2\\
&+\ln\left(\frac{\lambda^2}{Q^2}\right)\sum_{n=3}^{\infty}\frac{2+n}{(2-n)(1-n)n}\left(\frac{\lambda^2}{Q^2}\right)^n +\\
&\left.\left.+\sum_{n=3}^{\infty}\frac{-4+12n-3n^2-2n^3}{(2-n)^2(1-n)^2n^2}\left(\frac{\lambda^2}{Q^2}\right)^n\right]\right\}. \qquad \text{(C.18)}\end{aligned}$$

The first few terms are ($a_s \equiv \alpha_s/4\pi$)

$$\begin{aligned}\mathcal{F}_{1,R}(Q^2) \sim -\, a_s C_F \Bigg[&\ln^2\left(\frac{\lambda^2}{Q^2}\right) + 3\ln\left(\frac{\lambda^2}{Q^2}\right) + \frac{7}{2} + \frac{2\pi^2}{3}\\
&-2\left(\frac{\lambda^2}{Q^2}\right)\left(\ln^2\left(\frac{\lambda^2}{Q^2}\right) + 2 + \frac{2\pi^2}{3}\right) + \ldots\Bigg], \qquad \text{(C.19)}\end{aligned}$$

where the non-analytic logarithmic term in the correction which vanishes for $\lambda \to 0$ signals [25] the leading renormalon in the quark form factor.

It is easy to prove that the asymptotic expansion (C.18) is in fact an *exact* result since all sums in (C.18) are convergent in our limit (notice that they can be easily expressed in terms of usual functions after decomposition into partial fractions) and because there are no exponentially suppressed terms.

Indeed, one finds

$$\sum_{n=3}^{\infty}\frac{-4+12n-3n^2-2n^3}{(2-n)^2(1-n)^2n^2}\left(\frac{\lambda^2}{Q^2}\right)^n = \frac{\lambda^2}{Q^2} - \frac{11}{4}\left(\frac{\lambda^2}{Q^2}\right)^2 + \left[-1+\left(3-2\frac{\lambda^2}{Q^2}\right)\frac{\lambda^2}{Q^2}\right]\mathrm{Li}_2\left(\frac{\lambda^2}{Q^2}\right) \qquad \text{(C.20)}$$

and similar results for the other sums in (C.18).

Moreover the absence of exponentially suppressed terms is due to the fact that the asymptotic remainder integrals tend to zero. Indeed, choosing $T = \frac{2j+1}{2}$ where $j \in \mathbb{N}$, we have[16]

$$\left|\int_{-T-iT}^{-T+iT} ds \left(\frac{\lambda^2}{Q^2}\right)^{-s}\left(\frac{\pi}{\sin(\pi s)}\right)^2 \frac{1}{(2+s)(1+s)}\right| \leq 2T\left|\frac{\lambda^2}{Q^2}\right|^T \pi^2 \left|\frac{1}{(2-T)(1-T)}\right|, \qquad \text{(C.21)}$$

[16]This inequality follows from the fact that $\left|\int_{\mathcal{C}} ds f(s)\right| \leq ML$, where M is the maximum modulus of $f(s)$ on $\mathcal{C}$ and L is the length of $\mathcal{C}$.

and the righthand side vanishes for $T \to +\infty$ if $\frac{\lambda^2}{Q^2} < 1$. A similar result for the other integral of (C.14) is easily obtained.

After simplification of (C.18), one then has, because of the absence of exponentially suppressed terms, the final exact result

$$\mathcal{F}_{1,R}(Q^2) = a_s C_F \Bigg\{ \left(1-\frac{\lambda^2}{Q^2}\right)^2 \left[2\,\mathrm{Li}_2\left(\frac{\lambda^2}{Q^2}\right) + 2\ln\left(\frac{\lambda^2}{Q^2}\right)\ln\left(1-\frac{\lambda^2}{Q^2}\right) - \ln^2\left(\frac{\lambda^2}{Q^2}\right) - \frac{2\pi^2}{3}\right] - \frac{7}{2} + 2\frac{\lambda^2}{Q^2} - \ln\left(\frac{\lambda^2}{Q^2}\right)\left[3 - 2\frac{\lambda^2}{Q^2}\right]\Bigg\}, \tag{C.22}$$

which is indeed equal to $a_s C_F 2\mathcal{V}_s(\lambda^2/Q^2) = C_F \mathcal{V}_s(\lambda^2/Q^2)\frac{\alpha_s}{2\pi}$ (eq. (5.23)), with $\mathcal{V}_s(\lambda^2/Q^2)$ as defined in eq. (5.7). Our result agrees with [24], but only provided we interpret their "total correction to the renormalized hard vertex" as *twice* the one-loop renormalized quark form factor, since the normalization of $\mathcal{V}_s(\lambda^2/Q^2)$ in this latter reference is twice the one used in eq. (5.7). This factor of 2 arises because it is the *square* of the form factor which occurs in eq. (5.20).

Note added in proofs: the referee has pointed out to us that the all orders validity of eq. (2.37), hence of the conjecture eq. (2.26) in the DIS case, can actually be derived from the results of section 4 in the paper JHEP 0701, 076 (2007) by Becher, Neubert and Pecjak, once one notices that the matching function $C_V(Q^2, \mu)$ in this reference is related to $G(Q^2/\mu^2, a_s)$ by $G(Q^2/\mu^2, a_s) = \frac{d}{d\ln Q}\ln C_V(Q^2,\mu)$. We thank the referee for this valuable information.

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